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A quasi-one-dimensional asymptotic theory for non-linear water waves

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Abstract. Quasi-one-dimensional generalizations of different forms of the one-dimensional Boussinesq equations are derived asymptotically, then, from these quasi-one-dimensional Boussinesq equations, a consistent and significant second-order KP equation is derived, according to the Kadomtsev-Petviashvili [1] limiting process, the asymptotic expansions in the derivation of the non-linear Schrödinger–Poisson (NLSP) system of two equations, obtained by Freeman and Davey [2], in the long-water-waves limit are also determined.

Finally, I elucidate the influence of a bottom topography on the Boussinesq and KP equations.

1. Introduction

The main objective of this paper is to derive asymptotically various '*quasi-one-dimensional*' model equations for non-linear water waves.

There is always a semantic confusion with respect to counting dimensions in the non-linear water-waves problem. The non-linear water waves of interest here have *three*-dimensional velocity potentials $\phi(x, y, z, t)$ but *two*-dimensional free surfaces $z = \zeta(x, y, t)$. In this paper I consider the non-linear surface water waves of a finite (but *small*) amplitude in a channel with an *uneven* bottom and I suppose that their evolutions are *weakly* two-dimensional.

In fact, I derive, for these *nearly one-dimensional* surface water waves, some quasi-one-dimensional model equations. Naturally, in this case, in all these model equations x and y have *unequal status*. It is assumed that there will be a basic non-linear structure moving in the x-direction, whereas modulations will be in the x, y plane (Infeld [3]).

The wave motion, under force of gravity, of moving body of water, have a free surface in a channel with an uneven bottom, is one of the most interesting and successful applications of Nonlinear Hydrodynamics. Studies on water waves have always been enriched by interest shown among diverse fields of science, including Applied Mathematics and Singular Perturbation Techniques. Most of the perturbation work presented here is closely connected with the general presentation by Zakharov, Calogero and Eckhaus. (See Zakharov and Kuznetsov [4], Calogero and Eckhaus [5, 6], and also Calogero and Maccari [7].)

Given that the wave motion of an inviscid and incompressible fluid (such as water) is irrotational, then for ϕ it would be the obvious choice to derive the classical Laplace equation. However, Laplace's (elliptic) equation has little to do with waves and this reaction would be wrong (see, Whitham ([8], Chap. 13)) because of curious effects of the free surface conditions. Indeed, one boundary condition is given for Laplace's equation, but that is when the boundary is known (the so-called 'classical' Dirichlet or Neumann problems). Namely, two conditions are needed at a free (unknown) surface, $z = \zeta$, because the surface position, ζ, has to be determined as well as ϕ.

Furthermore, whereas Laplace's equation is linear, the two boundary free surface conditions are unfortunately *non*-linear. In Section 2 I shall give a full mathematical formulation for the water-waves problem.

However, it is necessary to note that in the presence of a free surface, the vorticity of an inviscid incompressible body of water does not necessarily remain zero if it is zero initially! Indeed, the free surface can intersect itself, as happens when a *wave breaks* and *vortex sheets* are formed. In this case, instead of Laplace's equation, it is necessary to consider the full Euler equations (always the case for an incompressible fluid). I do not consider here this important, but very difficult, question but analyze only the classical non-linear problem for ϕ and ζ, when the effect of the surface tension is included.

In this paper the derivation of model equations is given in depth, with rational uses of asymptotic methods. Indeed, it is important to understand that, in some cases, the establishment of these model equations is intuitive and heuristic and in fact it is not clear how to insert the model equation under consideration into a hierarchy of rational approximations, which in turn result consistently from the exact formulation of the water waves problem given in Section 2. Here, I note only two examples: the first one is the derivation of a *second-order KP equation* in Section 4 and the second is directly related to the asymptotic expansions for the derivation of the NLS–Poisson system of two equations in the long waves limit in Section 5, I specify the validity of asymptotic expansions for ϕ and ζ, relative to a small parameter and I derive for this a *new complementary equation* associated with the NLS-Poisson system of two equations.

The 'history' of theoretical research on the waves of a water surface was initiated by Russell's [9] discovery of the *solitary wave* phenomenon. His description of the wave as a solitary elevation of finite amplitude and permanent form was in contradiction with Airy's [10] *shallow water* theory prediction [for this shallow water theory see the books by Crapper ([11], Chap. 7) and Mei ([12], Chap. 11)]. The conflict between Russell's observations and Airy's shallow water theory (and also the classical Stokes' [13] expansion) was resolved independently by Boussinesq [14–16] and Rayleigh [17], (see, Miles [18] review paper). I specify only here that the Ursell [19] criterion; that there be a *balance between non-linearity and dispersion*, is an essential quality of the solitary wave. Rayleigh's [17] derivation of the profile of the solitary wave is reproduced by Lamb ([20], Section 252). It is more direct, but less penetrating than that of Boussinesq. The so-called '*Boussinesq equations*' are evolution equations for free surface displacement and mean horizontal velocity and are not restricted to unidirectional propagation (for a historical discussion, see Miles [18]). From these Boussinesq equations it is possible to derive the famous *KdV equation* (Korteweg and deVries [21]), invoking the prior assumption of *unilateral* propagation.

Interest waned after the resolution of the Airy–Stokes paradox by Boussinesq, Rayleigh and the appearance of the KdV equation and was sporadic prior to Zabusky and Kruskal's [22] discovery that: a *solitary wave typically dominates the asymptotic solution of the KdV equation*. Current interest stems from that discovery and is intense (so-called '*solitons dynamics*' and I mention in this way only four books by: Newell [23], Dodd et al. [24], Drazin and Johnson [25] and Infeld and Rowlands [26]). The theory of solitons is attractive and exciting. It brings together many branches of mathematics, some of which touch on deeper ideas and some of its aspects are both amazing and beautiful. But here, in this review paper, I do not touch on this problem and mention only the following major topics of this theory: the conservation laws and the Miura transformation, the inverse scattering transform, the Lax equation, the Bäcklund transformation, and the Hirota method.

Naturally, for cases in which non-linear surface waves in weakly dispersing shallow water are *not strictly one-dimensional*, the KdV equation no longer applies! In fact, it is necessary to derive a new approximate model equation for these cases – the so-called *KP equation*. (For a formal consistent derivation of this KP equation, see the paper by Freeman and Davey [2].)

Now, it ought to be noted that the inverse method and the structure of the KdV equation would have remained a mere mathematical curiosity, if further important model equations, those for water waves, had not been found that were solvable in this way. However, in 1972, in a paper of fundamental importance, Zakharov and Shabat [27] showed that the *non-linear Schrödinger (NSL) equation* could also be solved by the inverse method for initial data which decayed sufficiently quickly when $|x| \to \infty$ (see also: Zakharov and Kuznetsov [4] and Calogero and Maccari [7]).

For the water-waves problem the NLS equation was derived first, for *finite* depth, by Hasimoto and Ono [28], but a similar NLS equation had been deduced earlier, for an *infinite* depth, by Zakharov [29]. (For a theory of *deep-water waves* see the review paper by Yuen and Lake [30].)

For two-dimensional surface water waves, for finite depth and flat bottom, instead of the single NLS equation, Benney and Roskes [31] and Davey and Stewartson [32], derive a *system of two equations – the NLS–Poisson equations* (see, for instance Mei ([12], Chap. 12). When surface tension, is taken into account, expressions for the various constant coefficients, in the NLS–Poisson system, are given by Djordjevic and Redekopp [33] and Ablowitz and Segur [34] (see also Craik ([35], Chap. 6).

For an *uneven bottom* of the channel it is also possible to derive the Boussinesq, KdV and KP equations (see for instance: Peregrine [36], Ono [37], Johnson [38], Rosales and Papanicolaou [39], Xue-Nong Chen [40], Levi [41] and Benilov [42]). It is interesting to note that a KdV soliton travelling from one constant depth to another constant but smaller depth, *disintegrates* into several solitons of varying size, trailed by an oscillatory tail. This *'fission'* is clearly related to the result of the inverse scattering method (see Gardner et al. [43]) and the 'perturbed Ono–Johnson KdV' equation predicts the soliton fission that occurs as a solitary wave moves into a shelving region (Madsen and Mei [44]). Finally, concerning the *soliton interactions in two dimensions*, I mention the review paper by Freeman [45].

In Section 2 I give a brief mathematical formulation of the water waves problem and derive the relevant dimensionless exact Laplace equation and boundary conditions on the free surface and on the uneven bottom, with various non-dimensional parameters. In Section 3, I derive the quasi-one-dimensional generalized Boussinesq equations and from these Q1DGB-equations obtain the various forms of the quasi-one-dimensional Boussinesq (Q1DB)-equations. The second-order KP equation is derived in Section 4, that is, I obtain again the KP equation and the linear inhomogeneous associated equation describing the second order KP approximation. In Section 5, I derive the NLSP system of two equations in the long wave limit and elucidate the correspondence between KP and the Davey–Stewartson GNLSP equations. I also specify, by a rational use of the asymptotic method, the

Freeman–Davey expansions and derive a complementary equation associated with the NLSP system of two equations. Finally, Section 6 is devoted to a study of the influence of bottom topography in the Boussinesq and KP equations.

2. Governing equations and boundary conditions

2.1. Starting problem

The classical non-linear water waves problem is to find the irrotational motion of an inviscid, incompressible, homogeneous fluid, subject to the force of the gravity. The fluids rests on a horizontal and impermeable bottom of infinite extent at $z = -h_0$ (where h_0 is supposed finite) and has a free surface at $z = \zeta(x, y, t)$. A Cartesian coordinate system (x, y, z) is adopted, with $z = 0$ the position of the undisturbed two-dimensional free surface and the z-axis positive upwards.

For the above simple problem, the following *flat bottom boundary condition*:

$$w = \frac{\partial \phi}{\partial z} = 0, \quad \text{on } z = -h_0, \tag{1}$$

must be imposed.

The fluid velocity $\mathbf{v} = (u, v, w)$ is expressed by the gradient of a velocity potential $\phi(x, y, z, t)$, and for the function ϕ it is necessary to resolve the classical *Laplace equation*:

$$\nabla^2 \phi \equiv \Delta \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0, \tag{2}$$

with $\mathbf{v} = \nabla \phi$, when $-h_0 < z < \zeta(x, y, t)$.

The exact *kinematic boundary condition* (on the free surface) can be derived most readily by requiring that the substantial derivative of the quantity, $f = z - \zeta$ vanish on the free surface. The result of this condition is that:

$$\frac{\partial \phi}{\partial z} = \frac{\partial \zeta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \zeta}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \zeta}{\partial y}, \quad \text{on } z = \zeta(x, y, t). \tag{3}$$

The second boundary condition is the *dynamic condition* (on the free surface) and it is obtained from the Bernoulli classical integral for an unsteady potential flow. If it is assumed that the atmospheric pressure p_a is independent of position on the free surface, and if the constant of integration $C(t)$ (in the Bernoulli's integral) is suitably chosen, then the exact condition to be satisfied on the free surface is the following:

$$\frac{\partial \phi}{\partial t} + gz + \frac{1}{2}|\nabla \phi|^2 = 0, \quad \text{on } z = \zeta(x, y, t), \tag{4}$$

but only when we ignore the influence of surface tension.

The Laplace equation (2), with the three boundary conditions (1), (3) and (4), is our *classical three-dimensional non-linear water waves problem* (see for instance, Whitham ([8], Chap. 13)).

For the above problem, we consider an initial-value problem with the initial conditions:

$$\zeta = a_0 \zeta^0\left(\frac{x}{\lambda_0}, \frac{y}{\mu_0}\right) \quad \text{and} \quad \phi(x, y, z, 0) = 0 \,, \quad \text{for } t = 0 \,, \tag{5}$$

where λ_0 and μ_0 are the characteristic wavelengths (in the x and y directions) for our three-dimensional water wave motion and a_0 is the characteristic amplitude for the elevation of the free surface.

If the influence of surface tension is taken into account on the free surface, then in lieu of the dynamic condition (4) it is necessary to write the following *more complete dynamic* boundary condition:

$$\begin{aligned} \frac{\partial \phi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right] + gz = \frac{\tau_0}{\rho_0}\left[1 + \left(\frac{\partial \zeta}{\partial x}\right)^2 + \left(\frac{\partial \zeta}{\partial y}\right)^2\right]^{-3/2} \\ \times \left\{\left[1 + \left(\frac{\partial \zeta}{\partial x}\right)^2\right]\frac{\partial^2 \zeta}{\partial y^2} - 2\frac{\partial \zeta}{\partial x}\frac{\partial \zeta}{\partial y}\frac{\partial^2 \zeta}{\partial x\, \partial y} + \left[1 + \left(\frac{\partial \zeta}{\partial y}\right)^2\right]\frac{\partial^2 \zeta}{\partial x^2}\right\}, \end{aligned} \tag{6}$$

along the free surface $z = \zeta(x, y, t)$, where τ_0 is the surface-tension (constant) coefficient and ρ_0 is the density of our incompressible water.

Finally, if we take into account the bottom topography then, instead of the simple condition (1), we must write the following more complete uneven bottom condition:

$$\frac{\partial \phi}{\partial z} = g_0\left[\frac{\partial \phi}{\partial x}\frac{\partial G}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial G}{\partial y}\right], \quad \text{on } z = -h(x/l, y/m) \,, \tag{7}$$

with $h\,(x/l,\ y/m) \equiv -h_0 + g_0 G(x/l,\ y/m)$, where g_0 is a typical elevation for the bottom topography ($g_0 = G(0, 0)$) and l and m are the scale lengths associated with the variations in the channel bottom (in the x and y directions).

Concerning the conditions with respect to x and y, as it is assumed that the fluid rests on a bottom of infinite extent, it is necessary to impose some behavior conditions at infinity in the x and y directions. In fact, usually it is sufficient to suppose that the wave motion is periodic in x and y.

2.2. Dimensionless problem

Now, the dimensionless independent variables (with the primes) x', y', z' and t' are defined by:

$$x' = x/\lambda_0 \,, \qquad y' = y/\mu_0 \,, \qquad z' = z/h_0 \,, \qquad t' = t/t_0 \,, \tag{8}$$

with $t_0 = \lambda_0/c_0$ and $c_0^2 = gh_0$; in this case the Strouhal number

$$S = \lambda_0/c_0 t_0 \equiv 1 \,.$$

I also scale the functions ϕ and ζ:

$$\phi' = \frac{\phi}{\varepsilon c_0 \lambda_0} \quad \text{and} \quad \zeta' = \frac{\zeta}{a_0} \,, \tag{9}$$

with

$$\varepsilon = a_0/h_0 . \tag{10}$$

When the primes are dropped for $\phi(x, y, z, t)$, the following dimensionless Laplace equation instead of (2) is derived:

$$\frac{\partial^2\phi(x, y, z, t)}{\partial z^2} + \delta^2 \frac{\partial^2 \phi}{\partial x^2} + \Delta^2 \frac{\partial^2 \phi}{\partial y^2} = 0 , \quad \text{when } -1 < z < \varepsilon\zeta(x, y, t) . \tag{11}$$

In the Laplace equation (11), the following two non-dimensional parameters appear:

$$\delta = \frac{h_0}{\lambda_0}, \qquad \Delta = \frac{h_0}{\mu_0} . \tag{12}$$

Instead of (1) we have

$$\frac{\partial \phi}{\partial z} = 0 , \quad \text{on } z = -1 . \tag{13}$$

Instead of boundary conditions (3) and (4), on $z = \varepsilon\zeta(x, y, t)$, we find:

$$\frac{\partial \phi}{\partial z} = \delta^2 \frac{\partial \zeta}{\partial t} + \varepsilon \left[\delta^2 \frac{\partial \phi}{\partial x}\frac{\partial \zeta}{\partial x} + \Delta^2 \frac{\partial \phi}{\partial y}\frac{\partial \zeta}{\partial y} \right] \tag{14}$$

and

$$\frac{\partial \phi}{\partial t} + \zeta + \frac{\varepsilon}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\Delta}{\delta}\right)^2 \left(\frac{\partial \phi}{\partial y}\right)^2 + \frac{1}{\delta^2}\left(\frac{\partial \phi}{\partial z}\right)^2 \right], \tag{15}$$

along the free surface $z = \varepsilon\zeta(x, y, t)$.

Concerning the more complete boundary condition (6) we may obtain the following expression (instead of (15)):

$$\begin{aligned} &\frac{\partial \phi}{\partial t} + \zeta + \frac{\varepsilon}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\Delta}{\delta}\right)^2 \left(\frac{\partial \phi}{\partial y}\right)^2 + \frac{1}{\delta^2}\left(\frac{\partial \phi}{\partial z}\right)^2 \right] \\ &\quad = \delta^2\, \mathrm{We} \left[1 + \varepsilon^2\delta^2 \left(\frac{\partial \zeta}{\partial x}\right)^2 + \varepsilon^2\Delta^2 \left(\frac{\partial \zeta}{\partial y}\right)^2 \right]^{-3/2} \left\{ \left[1 + \varepsilon^2\Delta^2\left(\frac{\partial \zeta}{\partial y}\right)^2 \right] \frac{\partial^2 \zeta}{\partial x^2} - 2\varepsilon^2\Delta^2 \frac{\partial \zeta}{\partial x}\frac{\partial \zeta}{\partial y}\frac{\partial^2 \zeta}{\partial x\, \partial y} \right. \\ &\qquad \left. + \frac{\Delta^2}{\delta^2}\left[1 + \varepsilon^2\delta^2\left(\frac{\partial \zeta}{\partial x}\right)^2 \right] \frac{\partial^2 \zeta}{\partial y^2} \right\} , \quad \text{on } z = \varepsilon\zeta(x, y, t) , \end{aligned} \tag{16}$$

where the non-dimensional parameter

$$\mathrm{We} = \frac{\tau_0}{g\rho_0 h_0^2} \tag{17}$$

is the *Bond–Weber number*.

Finally, instead of the uneven bottom condition (7) the following dimensionless boundary condition is obtained:

$$\frac{\partial \phi}{\partial z} = \alpha \left[\delta^2 \frac{\partial \phi}{\partial x}\frac{\partial G}{\partial x} + \Delta^2 \frac{\partial \phi}{\partial y}\frac{\partial G}{\partial y} \right] , \quad \text{on } z = -1 + \alpha G(\hat{x}, \hat{y}) , \tag{18}$$

with the following three non-dimensional parameters:

$$\alpha = g_0/h_0\,, \qquad \beta = \lambda_0/l\,, \qquad \gamma = \mu_0/m\,, \tag{19}$$

and with the uneven bottom horizontal variables:

$$\hat{x} = \beta x\,, \qquad \hat{y} = \gamma y\,. \tag{20}$$

In the above dimensionless problem: (11), (13), (14) and (15), the parameter $\varepsilon = a_0/h_0$ is the *non-linearity* parameter and for $\varepsilon \to 0$, with x, y, z and t fixed, and also for fixed values of δ and Δ, instead of (11), (13), (14) and (15) the classical *linear* water-waves problem for ϕ is obtained:

$$\begin{aligned}
&\frac{\partial^2\phi}{\partial z^2} + \delta^2\frac{\partial^2\phi}{\partial x^2} + \Delta^2\frac{\partial^2\phi}{\partial y^2} = 0\,, && -1 < z < 0\,;\\
&\frac{\partial\phi}{\partial z} = 0\,, && \text{on } z = -1\,;\\
&\frac{\partial\phi}{\partial z} + \delta^2\frac{\partial^2\phi}{\partial t^2} = 0\,, && \text{on } z = 0\,.
\end{aligned} \tag{21}$$

The parameter δ is the *long longitudinal*, (x direction), water-wave parameter and Δ is the *long transversal*, (y direction), water-wave parameter.

In Section 3, we mainly consider the following asymptotic situation:

$$\varepsilon \ll 1\,, \qquad \delta \ll 1 \quad \text{and} \quad \Delta \ll 1\,, \tag{22}$$

with two similarity relations:

$$\delta^2 = \kappa_0\varepsilon \quad \text{and} \quad \Delta = \nu_0\varepsilon\,, \tag{23}$$

where κ_0 and ν_0 are of the order one when $\varepsilon \to 0$.

In fact, we assume that:

(a) water-wave amplitudes are *small*;
(b) the water layer is *shallow* relative to typical horizontal wavelengths;
(c) the water waves are *nearly* one-dimensional; and
(d) these above three small effects all have *comparable influence* (all three effects balance, according to the Ursell [19] criterion).

When the parameter $\alpha \ll 1$, we have a *small effect of the elevation of* the uneven bottom topography. Finally, for $\beta \gg 1$ and for $\gamma \gg 1$, we have a *rough* bottom and for $\beta \ll 1$ and $\gamma \ll 1$, a *slowly varying* bottom.

If we consider now the more complete dynamic boundary condition (16), then it will be necessary to consider two cases: in the first case it is supposed that $\mathrm{We} = O(1)$, is of the order one and in this case, in the linear problem (21), the last boundary condition (on $z = 0$) must be replaced by the following condition:

$$\delta^2\frac{\partial^2\phi}{\partial t^2} + \frac{\partial\phi}{\partial z} + \mathrm{We}\frac{\partial^3\phi}{\partial z^3} = 0\,, \quad \text{on } z = 0\,; \tag{24}$$

in the second case it is assumed that

$$\text{We} \gg 1 \quad \text{but} \quad \delta^2 \text{We} = \hat{T} = O(1) \tag{25}$$

and this last case is very significant in the framework of the Q1DGB-equations (see Section 3). But from these Q1DGB-equations it is not easy to derive a 'unidirectional' limiting equation similar to the KP equation.

3. Boussinesq equations

3.1. Quasi-one-dimensional generalized Boussinesq (Q1DGB) equations

First, for the derivation of the so-called Q1DGB equations, let us consider here the following dimensionless problem according to (2.11), (2.13), (2.14) and (2.16)*:

$$\frac{\partial^2\phi}{\partial z^2} + \delta^2\frac{\partial^2\phi}{\partial x^2} + \Delta^2\frac{\partial^2\phi}{\partial y^2} = 0\,, \quad -1 < z < \varepsilon\zeta(x, y, t)\,; \tag{1a}$$

$$\frac{\partial\phi}{\partial z} = 0\,, \quad \text{on } z = -1\,; \tag{1b}$$

$$\frac{\partial\phi}{\partial z} = \delta^2\frac{\partial\zeta}{\partial t} + \varepsilon\delta^2\frac{\partial\phi}{\partial x}\frac{\partial\zeta}{\partial x} + \varepsilon\Delta^2\frac{\partial\phi}{\partial y}\frac{\partial\zeta}{\partial y}\,, \quad \text{on } z = \varepsilon\zeta(x, y, t); \tag{1c}$$

$$\frac{\partial\phi}{\partial t} + \zeta + \frac{\varepsilon}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\Delta}{\delta}\right)^2\left(\frac{\partial\phi}{\partial y}\right)^2 + \frac{1}{\delta^2}\left(\frac{\partial\phi}{\partial z}\right)^2\right]$$
$$= \delta^2\,\text{We}\left\{\frac{\partial^2\zeta}{\partial x^2} + \left(\frac{\Delta}{\delta}\right)^2\frac{\partial^2\zeta}{\partial y^2} - \frac{3}{2}\varepsilon^2\delta^2\left(\frac{\partial\zeta}{\partial x}\right)^2\frac{\partial^2\zeta}{\partial x^2} + O(\varepsilon^2\Delta^2)\right\}, \quad \text{on } z = \varepsilon\zeta(x, y, t)\,. \tag{1d}$$

Laplace's equation (1a) is the only equation which contains z in its solution and this variation may be made explicit by formally expanding its solution in powers of δ^2 and Δ^2 and writing:

$$\phi = \phi_{00} + \delta^2\phi_{10} + \delta^4\phi_{20} + \Delta^2\phi_{01} + \delta^6\phi_{30} + \delta^2\Delta^2\phi_{11} + \cdots\,. \tag{2}$$

This above asymptotic representation is consistent with our main hypothesis (2.22) and (2.23).

For ϕ_{00} it is necessary to solve the following trivial problem:

$$\frac{\partial^2\phi_{00}}{\partial z^2} = 0 \quad \text{and} \quad \frac{\partial\phi_{00}}{\partial z} = 0\,, \quad \text{on } z = -1 \text{ and } z = 0\,,$$

and the solution is simply:

$$\phi_{00} \equiv F(x, y, t)\,. \tag{3}$$

Below, for simplicity, let us assume that: *$F(x, y, t)$ is the value of ϕ (in (2)) on $z = -1$*, and in this case:

$$\phi_{10} = \phi_{20} = \phi_{01} = \phi_{30} = \phi_{11} = \cdots = 0\,, \quad \text{on } z = -1\,. \tag{4}$$

But, according to (1b),

$$\frac{\partial\phi_{10}}{\partial z}=\frac{\partial\phi_{20}}{\partial z}=\frac{\partial\phi_{01}}{\partial z}=\frac{\partial\phi_{30}}{\partial z}=\frac{\partial\phi_{11}}{\partial z}=\cdots=0\,,\quad \text{on } z=-1\,, \tag{5}$$

and as a consequence of (2) and (3) we may write immediately the solution for ϕ_{10}, ϕ_{20}, ϕ_{01}, ϕ_{30} and ϕ_{11} in the following explicit forms:

$$\left.\begin{aligned}
&\phi_{10}=-\frac{1}{2}\frac{\partial^2 F}{\partial x^2}(z+1)^2\,; \qquad \phi_{20}=\frac{1}{24}\frac{\partial^4 F}{\partial x^4}(z+1)^4\,;\\
&\phi_{01}=-\frac{1}{2}\frac{\partial^2 F}{\partial y^2}(z+1)^2\,; \qquad \phi_{30}=-\frac{1}{720}\frac{\partial^6 F}{\partial x^6}(z+1)^6\,;\\
&\phi_{11}=\frac{1}{24}\frac{\partial^4 F}{\partial x^2\,\partial y^2}(z+1)^4\,.
\end{aligned}\right\} \tag{6}$$

Finally, we obtain, instead of (2), the following asymptotic representation for ϕ, as solution of the Laplace's equation with the bottom condition, on $z=-1$,

$$\begin{aligned}
\phi(x,y,z,t)=F(x,y,t)&-\frac{\delta^2}{2}(z+1)^2\frac{\partial^2 F}{\partial x^2}+\delta^4\frac{(z+1)^4}{24}\frac{\partial^4 F}{\partial x^4}-\frac{\Delta^2}{2}(z+1)^2\frac{\partial^2 F}{\partial y^2}\\
&-\delta^6\frac{(z+1)^6}{720}\frac{\partial^6 F}{\partial x^6}+\delta^2\Delta^2\frac{(z+1)^4}{24}\frac{\partial^4 F}{\partial x^2\,\partial y^2}+\cdots\,.
\end{aligned} \tag{7}$$

Now, by means of Taylor expansions, it is possible to calculate the derivatives: $\partial\phi/\partial t$, $\partial\phi/\partial x$, $\partial\phi/\partial y$, and $\partial\phi/\partial z$ (on $z=\varepsilon\zeta(x,y,t)$), but *for* $z=0$.

Finally, if we take into account our two boundary conditions on the free surface $z=\varepsilon\zeta(x,y,t)$ (the two conditions (1c) and (1d)), with the relations for the derivatives on $z=0$, and also the three similarity relations (2.23) and (2.25) we obtain for the two unknown functions $F(x,y,t)$ and $\zeta(x,y,t)$ the following two approximate equations:

$$\begin{aligned}
&\frac{\partial F}{\partial t}+\zeta-\hat{T}\frac{\partial^2\zeta}{\partial x^2}+\varepsilon\left\{\frac{1}{2}\left(\frac{\partial F}{\partial x}\right)^2-\frac{\kappa_0}{2}\frac{\partial^3 F}{\partial t\,\partial x^2}-\hat{T}\frac{\nu_0^2}{\kappa_0}\frac{\partial^2\zeta}{\partial y^2}\right\}\\
&\quad+\varepsilon^2\left\{\frac{\kappa_0^2}{4}\frac{\partial^5 F}{\partial t\,\partial x^4}+\frac{\kappa_0}{2}\left(\frac{\partial^2 F}{\partial x^2}\right)^2-\frac{\nu_0^2}{2}\frac{\partial^3 F}{\partial t\,\partial y^2}+\frac{\nu_0^2}{2\kappa_0}\left(\frac{\partial F}{\partial y}\right)^2-\frac{\kappa_0}{2}\frac{\partial F}{\partial x}\frac{\partial^3 F}{\partial x^3}\right\}\\
&\quad-\kappa_0\frac{\partial}{\partial t}\left(\zeta\frac{\partial^2 F}{\partial x^2}\right)=O(\varepsilon^3)\,;
\end{aligned} \tag{8a}$$

and

$$\begin{aligned}
&\frac{\partial\zeta}{\partial t}+\frac{\partial^2 F}{\partial x^2}+\varepsilon\left\{\frac{\nu_0^2}{\kappa_0}\frac{\partial^2 F}{\partial y^2}+\frac{\partial}{\partial t}\left(\zeta\frac{\partial F}{\partial x}\right)-\frac{\kappa_0}{6}\frac{\partial^4 F}{\partial x^4}\right\}\\
&\quad+\varepsilon^2\left\{\frac{\kappa_0^2}{120}\frac{\partial^6 F}{\partial x^6}-\frac{\kappa_0}{2}\frac{\partial}{\partial x}\left(\zeta\frac{\partial^3 F}{\partial x^3}\right)+\frac{\nu_0^2}{\kappa_0}\frac{\partial}{\partial y}\left(\zeta\frac{\partial F}{\partial y}\right)-\frac{\nu_0^2}{6}\frac{\partial^4 F}{\partial x^2\,\partial y^2}\right\}=O(\varepsilon^3)\,,
\end{aligned} \tag{8b}$$

when $\hat{T}=O(1)$ and with an error of $O(\varepsilon^3)$. Here, equations (8a) and (8b), which include the terms of order $O(\varepsilon)$ and $O(\varepsilon^2)$, are called the *quasi-one-dimensional generalized Boussinesq (Q1DGB) equations.*

Naturally, in equations (8a,b) the unknown functions $\zeta(x, y, t)$ and $F(x, y, t)$ are implicit functions of ε and we can write:

$$\zeta = \zeta_0 + \varepsilon\zeta_1 + \varepsilon^2\zeta_2 + \cdots ; \qquad F = F_0 + \varepsilon F_1 + \varepsilon^2 F_2 + \cdots . \tag{9}$$

With (9), from equations (8a,b), we derive successively the following limiting equations for: (F_0, ζ_0), (F_1, ζ_1) and (F_2, ζ_2),

$$\frac{\partial F_0}{\partial t} + \zeta_0 - \hat{T}\frac{\partial^2 \zeta_0}{\partial x^2} = 0 , \qquad \frac{\partial \zeta_0}{\partial t} + \frac{\partial^2 F_0}{\partial x^2} = 0 ; \tag{10a}$$

$$\left.\begin{aligned} &\frac{\partial F_1}{\partial t} + \zeta_1 - \hat{T}\frac{\partial^2 \zeta_1}{\partial x^2} = \frac{\kappa_0}{2}\frac{\partial^3 F_0}{\partial t\,\partial x^2} + \hat{T}\frac{\nu_0^2}{\kappa_0}\frac{\partial^2 \zeta_0}{\partial y^2} - \frac{1}{2}\left(\frac{\partial F_0}{\partial x}\right)^2 , \\ &\frac{\partial \zeta_1}{\partial t} + \frac{\partial^2 F_1}{\partial x^2} = \frac{\kappa_0}{6}\frac{\partial^4 F_0}{\partial x^4} - \frac{\partial}{\partial x}\left(\zeta_0 \frac{\partial F_0}{\partial x}\right) - \frac{\nu_0^2}{\kappa_0}\frac{\partial^2 F_0}{\partial y^2} ; \end{aligned}\right\} \tag{10b}$$

and

$$\left.\begin{aligned} &\frac{\partial F_2}{\partial t} + \zeta_2 - \hat{T}\frac{\partial^2 \zeta_2}{\partial x^2} = \frac{\kappa_0}{2}\frac{\partial^3 F_1}{\partial t\,\partial x^2} + \hat{T}\frac{\nu_0^2}{\kappa_0}\frac{\partial^2 \zeta_1}{\partial y^2} - \frac{\partial F_0}{\partial x}\frac{\partial F_1}{\partial x} - \frac{\kappa_0^2}{24}\frac{\partial^5 F_0}{\partial t\,\partial x^4} - \frac{\kappa_0}{2}\left(\frac{\partial^2 F_0}{\partial x^2}\right)^2 \\ &\qquad + \frac{\nu_0^2}{2}\frac{\partial^3 F_0}{\partial t\,\partial y^2} - \frac{\nu_0^2}{2\kappa_0}\left(\frac{\partial F_0}{\partial y}\right)^2 + \frac{\kappa_0}{2}\frac{\partial^3 F_0}{\partial x^3}\frac{\partial F_0}{\partial x} + \kappa_0\frac{\partial}{\partial t}\left(\zeta_0\frac{\partial^2 F_0}{\partial x^2}\right) , \\ &\frac{\partial \zeta_2}{\partial t} + \frac{\partial^2 F_2}{\partial x^2} = \frac{\kappa_0}{6}\frac{\partial^4 F_1}{\partial x^4} - \frac{\nu_0^2}{\kappa_0}\frac{\partial^2 F_1}{\partial y^2} + \frac{\nu_0^2}{6}\frac{\partial^4 F_0}{\partial x^2\,\partial y^2} - \frac{\partial}{\partial x}\left(\zeta_0\frac{\partial F_1}{\partial x} + \frac{\partial F_0}{\partial x}\zeta_1\right) - \frac{\kappa_0^2}{120}\frac{\partial^6 F_0}{\partial x^6} \\ &\qquad + \frac{\kappa_0}{6}\frac{\partial}{\partial x}\left(\zeta_0\frac{\partial^3 F_0}{\partial x^3}\right) - \frac{\nu_0^2}{\kappa_0}\frac{\partial}{\partial y}\left(\zeta_0\frac{\partial F_0}{\partial y}\right) . \end{aligned}\right\} \tag{10c}$$

3.2. Quasi-one-dimensional Boussinesq (Q1DB) equations

Let us consider now equations (10a) and (10b) and assume that the Bond–Weber number $\mathrm{We} = O(1)$. In this case, according to (2.25), all the terms are proportional to:

$$\hat{T} = \kappa_0\,\mathrm{We}\,\varepsilon , \tag{11}$$

and are therefore of the order of ε.

Hence, instead of equations (10a) and (10b), we obtain the following system of two equations for F_0 and F_1:

$$\frac{\partial^2 F_0}{\partial t^2} - \frac{\partial^2 F_0}{\partial x^2} = 0 ; \tag{12}$$

$$\begin{aligned} \frac{\partial^2 F_1}{\partial t^2} - \frac{\partial^2 F_1}{\partial x^2} &= \frac{\kappa_0}{2}\frac{\partial^4 F_0}{\partial t^2\,\partial x^2} - \frac{\kappa_0}{6}\frac{\partial^4 F_0}{\partial x^4} + \frac{\nu_0^2}{\kappa_0}\frac{\partial^2 F_0}{\partial y^2} \\ &\quad - \kappa_0\,\mathrm{We}\frac{\partial^4 F_0}{\partial t^2\,\partial x^2} - \frac{\partial F_0}{\partial x}\frac{\partial}{\partial t}\left(\frac{\partial F_0}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial F_0}{\partial t}\frac{\partial F_0}{\partial x}\right) , \end{aligned} \tag{13}$$

after the elimination of functions ζ_0 and ζ_1. Now, if:

$$F^* = F_0 + \varepsilon F_1 \tag{14}$$

then, from (12) and (13), the following single Q1DB-equation for F^* is derived:

$$\frac{\partial^2 F^*}{\partial t^2} - \frac{\partial^2 F^*}{\partial x^2} - \varepsilon \frac{\nu_0^2}{\kappa_0} \frac{\partial^2 F^*}{\partial y^2} - \varepsilon \kappa_0 \left[\mathrm{We} - \frac{1}{3}\right] \frac{\partial^2 F^*}{\partial t^2 \partial x^2} + \varepsilon \frac{\partial}{\partial t}\left[\left(\frac{\partial F^*}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial F^*}{\partial t}\right)^2\right] = 0\,, \tag{15}$$

when we take into account that

$$\frac{\partial}{\partial x}\left(\frac{\partial F_0}{\partial t}\frac{\partial F_0}{\partial x}\right) = \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial F_0}{\partial t}\right)^2 + \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial F_0}{\partial x}\right)^2\,, \quad \text{since: } \frac{\partial^2 F_0}{\partial t^2} = \frac{\partial^2 F_0}{\partial x^2}\,.$$

The above Q1DB-equation (15) is also obtained directly from the start, problems (1a)–(1d), with the similarity relations (2.23), when we take into account, instead of the representation (7), the following 'more' complete representation for $\phi(x, y, z, t)$:

$$\begin{aligned} \phi = F_0(x, y, t) &+ \varepsilon\left[F_1(x, y, t) - \frac{\kappa_0}{2}(z+1)^2 \frac{\partial^2 F_0}{\partial x^2}\right] \\ &+ \varepsilon^2\left\{F_2(x, y, t) + \frac{\kappa_0^2}{24}(z+1)^4 \frac{\partial^4 F_0}{\partial x^4} - \frac{\kappa_0}{2}(z+1)^2 \frac{\partial^2 F_1}{\partial x^2} - \frac{\nu_0^2}{2}(z+1)^2 \frac{\partial^2 F_0}{\partial y^2}\right\} + O(\varepsilon^3)\,, \end{aligned} \tag{16}$$

where F_0, F_1, $F_2, \ldots$ are unknown functions of the independent variables x, y and t. Naturally, in this case, in (16), $F_0(x, y, t)$ is not the value of ϕ on $z = -1$!

Next, it is assumed that:

$$\begin{aligned} &F_j = F_{j0} + \varepsilon F_{j1} + \cdots, \quad j = 0, 1, 2, \ldots, \\ &\zeta = \zeta_0 + \varepsilon \zeta_1 + \cdots, \end{aligned} \tag{17}$$

and with (16) and (17) it is possible, again, to calculate the derivatives: $\partial\phi/\partial t$, $\partial\phi/\partial x$, $\partial\phi/\partial y$ and $\partial\phi/\partial z$ (on $z = \varepsilon\zeta(z, y, t)$), but for $z = 0$.

Finally, from the two boundary conditions (1c) and (1d), we derive for the functions (F_{00}, ζ_0), and $(F_{01} + F_{10} \equiv G_1, \zeta_1)$ a set of limiting equations (very similar to (12) and (13) after the elimination of ζ_0 and ζ_1) and from these two limiting equations the same Q1DB equation (15), but for the function: $F^{**} = F_{00} + \varepsilon G_1$, is derived.

At last, we can write the Q1DB-equation as a system of equations for the free surface position function $\zeta(x, y, t)$ and the horizontal velocity components: $u(x, y, t) = \partial F/\partial x$ and $v(x, y, t) = \partial F/\partial y$ in he following form (but for We = 0):

$$\left.\begin{aligned} &\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x}[(1 + \varepsilon\zeta)u] + \varepsilon\frac{\partial v}{\partial y} - \frac{\varepsilon}{6}\frac{\partial^3 u}{\partial x^3} = 0\,; \\ &\frac{\partial \zeta}{\partial x} + \frac{\partial u}{\partial t} + \varepsilon u \frac{\partial u}{\partial x} - \varepsilon\frac{\partial^3 u}{\partial t\, \partial x^2} = 0\,; \qquad \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}\,. \end{aligned}\right\} \tag{18}$$

Averaging

$$\frac{\partial\phi}{\partial x} = u - \varepsilon\frac{z^2}{2}\frac{\partial^2 u}{\partial x^2} + O(\varepsilon^2)$$

and

$$\frac{\partial\phi}{\partial y} = v - \varepsilon\frac{z^2}{2}\frac{\partial^2 v}{\partial x^2} + O(\varepsilon^2)\,,$$

over the depth yields for u:

$$u = U + \frac{\varepsilon}{6}\frac{\partial^2 U}{\partial x^2} + O(\varepsilon^2)\,, \tag{19a}$$

and for v

$$v = V + \frac{\varepsilon}{6}\frac{\partial^2 V}{\partial x^2} + O(\varepsilon^2)\,. \tag{19b}$$

When (19a,b) are used in the Q1DB-equations (18), we obtain the following form for our Q1DB-equations:

$$\left.\begin{aligned}
&\frac{\partial\zeta}{\partial t} + \frac{\partial}{\partial x}[(1+\varepsilon\zeta)U] + \varepsilon\frac{\partial V}{\partial y} = 0\,;\\
&\frac{\partial\zeta}{\partial x} + \frac{\partial U}{\partial t} + \varepsilon U\frac{\partial U}{\partial x} - \frac{\varepsilon}{3}\frac{\partial^3 U}{\partial t\,\partial x^2} = 0\,;\\
&\frac{\partial V}{\partial x} = \frac{\partial U}{\partial y}\,,
\end{aligned}\right\} \tag{20}$$

for $U(x, y, t)$, $V(x, y, t)$ and $\zeta(x, y, t)$.

Note that our Q1DB-equations (20) are not similar to the three-dimensional generalization of the Boussinesq equations, derived by Infeld ([3]; Appendix 1 – BI equations). Seemingly these BI, Infeld, equations, are inconsistent from the point of view of asymptotic methodology. Instead of (20), we can also derive two equations for $\zeta(x, y, t)$ and $U(x, y, t)$, if we differentiate the first equation of (20) with respect to x and utilize the third equation of (20):

$$\left.\begin{aligned}
&\frac{\partial\zeta}{\partial t} + \frac{\partial U}{\partial t} + \varepsilon U\frac{\partial U}{\partial x} = \frac{\varepsilon}{3}\frac{\partial^3 U}{\partial t\,\partial x^2}\,;\\
&\frac{\partial}{\partial t}\left(\frac{\partial\zeta}{\partial x}\right) + \frac{\partial^2}{\partial x^2}[(1+\varepsilon\zeta)U] + \varepsilon\frac{\partial^2 U}{\partial y^2} = 0\,.
\end{aligned}\right\} \tag{21}$$

4. The KP limit

Let us now return to Q1DGB-equations (3.8a,b), for the functions $F(x, y, t)$ and $\zeta(x, y, t)$. We assume also that: $\hat{T} = \kappa_0\,\mathrm{We}\,\varepsilon$, with $\mathrm{We} = O(1)$.

In relation with the KP limiting process it is necessary to introduce a slow time scale: $\tau = \varepsilon t$ and in this case:

$$F = F(x, y, t, \tau; \varepsilon) \quad\text{and}\quad \zeta = \zeta(x, y, t, \tau; \varepsilon)\,, \tag{1}$$

with

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial t}+\varepsilon\frac{\partial}{\partial\tau}\,.$$

Then, instead of (3.8a) and (3.8b), for $F(x, y, t, \tau; \varepsilon)$ and $\zeta(x, y, t, \tau; \varepsilon)$ the following system of two equations are obtained:

$$\frac{\partial F}{\partial t}+\zeta+\varepsilon\left\{\frac{1}{2}\left(\frac{\partial F}{\partial x}\right)^2-\frac{\kappa_0}{2}\frac{\partial^3 F}{\partial t\,\partial x^2}+\frac{\partial F}{\partial\tau}-\kappa_0\,\mathrm{We}\frac{\partial^2\zeta}{\partial x^2}\right\}$$
$$+\varepsilon^2\left\{-\frac{\kappa_0}{2}\frac{\partial^3 F}{\partial\tau\,\partial x^2}+\frac{\kappa_0^2}{24}\frac{\partial^5 F}{\partial t\,\partial x^4}+\frac{\kappa_0}{2}\left(\frac{\partial^2 F}{\partial x^2}\right)^2-\frac{\kappa_0}{2}\frac{\partial F}{\partial x}\frac{\partial^3 F}{\partial x^3}\right.$$
$$\left.-\kappa_0\frac{\partial}{\partial t}\left(\zeta\frac{\partial^2 F}{\partial x^2}\right)+\frac{\nu_0^2}{2\kappa_0}\left(\frac{\partial F}{\partial y}\right)^2-\frac{\nu_0^2}{2}\frac{\partial^3 F}{\partial t\,\partial y^2}-\nu_0^2\,\mathrm{We}\frac{\partial^2\zeta}{\partial y^2}\right\}=O(\varepsilon^3)\,; \tag{2a}$$

and

$$\frac{\partial\zeta}{\partial t}+\frac{\partial^2 F}{\partial x^2}+\varepsilon\left\{-\frac{\kappa_0}{6}\frac{\partial^4 F}{\partial x^4}+\frac{\partial}{\partial x}\left(\zeta\frac{\partial F}{\partial x}\right)+\frac{\partial\zeta}{\partial\tau}+\frac{\nu_0^2}{\kappa_0}\frac{\partial^2 F}{\partial y^2}\right\}$$
$$+\varepsilon^2\left\{\frac{\kappa_0^2}{120}\frac{\partial^6 F}{\partial x^6}-\frac{\kappa_0}{6}\frac{\partial}{\partial x}\left(\zeta\frac{\partial^3 F}{\partial x^3}\right)+\frac{\nu_0^2}{\kappa_0}\frac{\partial}{\partial y}\left(\zeta\frac{\partial F}{\partial y}\right)-\frac{\nu_0^2}{6}\frac{\partial^4 F}{\partial x^2\,\partial y^2}\right\}=O(\varepsilon^3)\,. \tag{2b}$$

If the appropriate asymptotic expansions of F and ζ are:

$$F=F_0+\varepsilon F_1+\varepsilon^2 F_2+\cdots\,;\qquad \zeta=\zeta_0+\varepsilon\zeta_1+\varepsilon^2\zeta_2+\cdots\,, \tag{3}$$

then, as a result, the following set of equations to different powers of ε may be derived:

$$O(\varepsilon^0):\quad \frac{\partial F_0}{\partial t}+\zeta_0=0\,,\qquad \frac{\partial^2 F_0}{\partial x^2}+\frac{\partial\zeta_0}{\partial t}=0\,; \tag{4a}$$

$$O(\varepsilon^1):\begin{cases}\dfrac{\partial F_1}{\partial t}+\zeta_1+\dfrac{1}{2}\left(\dfrac{\partial F_0}{\partial x}\right)^2-\dfrac{\kappa_0}{2}\dfrac{\partial^3 F_0}{\partial t\,\partial x^2}-\kappa_0\,\mathrm{We}\dfrac{\partial^2\zeta_0}{\partial x^2}+\dfrac{\partial F_0}{\partial\tau}=0\,,\\[2ex] \dfrac{\partial^2 F_1}{\partial x^2}+\dfrac{\partial\zeta_1}{\partial t}-\dfrac{\kappa_0}{6}\dfrac{\partial^4 F_0}{\partial x^4}+\dfrac{\partial}{\partial x}\left(\zeta_0\dfrac{\partial F_0}{\partial x}\right)+\dfrac{\nu_0^2}{\kappa_0}\dfrac{\partial^2 F_0}{\partial y^2}+\dfrac{\partial\zeta_0}{\partial\tau}=0\,;\end{cases} \tag{4b}$$

and

$$O(\varepsilon^2):\begin{cases}\dfrac{\partial F_2}{\partial t}+\zeta_2+\dfrac{\partial F_0}{\partial x}\dfrac{\partial F_1}{\partial x}-\dfrac{\kappa_0}{2}\dfrac{\partial^3 F_1}{\partial t\,\partial x^2}+\dfrac{\partial F_1}{\partial\tau}-\kappa_0\,\mathrm{We}\dfrac{\partial^2\zeta_1}{\partial x^2}-\dfrac{\kappa_0}{2}\dfrac{\partial^3 F_0}{\partial\tau\,\partial x^2}+\dfrac{\kappa_0^2}{24}\dfrac{\partial^5 F_0}{\partial t\,\partial x^4}\\[2ex] \quad+\dfrac{\kappa_0}{2}\left(\dfrac{\partial^2 F_0}{\partial x^2}\right)^2-\dfrac{\kappa_0}{2}\dfrac{\partial^3 F_0}{\partial x^3}\dfrac{\partial F_0}{\partial x}-\kappa_0\dfrac{\partial}{\partial t}\left(\zeta_0\dfrac{\partial^2 F_0}{\partial x^2}\right)+\dfrac{\nu_0^2}{2\kappa_0}\left(\dfrac{\partial F_0}{\partial y}\right)^2\\[2ex] \quad-\dfrac{\nu_0^2}{2}\dfrac{\partial^3 F_0}{\partial t\,\partial y^2}-\nu_0^2\,\mathrm{We}\dfrac{\partial^2\zeta_0}{\partial y^2}=0\,,\\[2ex] \dfrac{\partial\zeta_2}{\partial t}+\dfrac{\partial^2 F_2}{\partial x^2}-\dfrac{\kappa_0}{6}\dfrac{\partial^4 F_1}{\partial x^4}+\dfrac{\nu_0^2}{\kappa_0}\dfrac{\partial^2 F_1}{\partial y^2}+\dfrac{\partial}{\partial x}\left(\zeta_0\dfrac{\partial F_1}{\partial x}+\dfrac{\partial F_0}{\partial x}\zeta_1\right)+\dfrac{\partial\zeta_1}{\partial\tau}\\[2ex] \quad+\dfrac{\kappa_0^2}{120}\dfrac{\partial^6 F_0}{\partial x^6}-\dfrac{\kappa_0}{6}\dfrac{\partial}{\partial x}\left(\zeta_0\dfrac{\partial^3 F_0}{\partial x^3}\right)+\dfrac{\nu_0^2}{\kappa_0}\dfrac{\partial}{\partial y}\left(\zeta_0\dfrac{\partial F_0}{\partial y}\right)-\dfrac{\nu_0^2}{6}\dfrac{\partial^4 F_0}{\partial x^2\,\partial y^2}=0\,.\end{cases} \tag{4c}$$

From (4a) it follows that F_0 (and ζ_0) depends on x and t either through $x-t$ or $x+t$. Here, we shall only consider the wave propagating to the right, and hence we shall assume that the dependence of F_0 (and ζ_0) on x and t is only through the variable $\xi = x - t$ and in this case:

$$\frac{\partial}{\partial x} \equiv \frac{\partial}{\partial \xi} \quad \text{and} \quad \frac{\partial}{\partial t} = -\frac{\partial}{\partial \xi}.$$

Furthermore, when F_1 (and ζ_1) depends only on ξ, τ and y, equation (4b) can be reduced to:

$$\left.\begin{aligned} \frac{\partial F_1}{\partial \xi} &= \zeta_1 + \frac{1}{2}\left(\frac{\partial F_0}{\partial \xi}\right)^2 + \frac{\kappa_0}{2}\frac{\partial^3 F_0}{\partial \xi^3} - \kappa_0 \,\mathrm{We}\frac{\partial^3 F_0}{\partial \xi^3} + \frac{\partial F_0}{\partial \tau}; \\ \frac{\partial \zeta_1}{\partial \xi} &= \frac{\partial^2 F_1}{\partial \xi^2} - \frac{\kappa_0}{6}\frac{\partial^4 F_0}{\partial \xi^4} + \frac{\partial}{\partial \xi}\left(\frac{\partial F_0}{\partial \xi}\right)^2 + \frac{\partial^2 F_0}{\partial \tau\,\partial \xi} + \frac{\nu_0^2}{\kappa_0}\frac{\partial^2 F_0}{\partial y^2}, \end{aligned}\right\} \tag{5}$$

and, as a consequence of (5), the classical KP equation, for $F_0(\xi, \tau, y)$, is derived:

$$\frac{\partial}{\partial \xi}\left\{2\frac{\partial F_0}{\partial \tau} + \frac{3}{2}\left(\frac{\partial F_0}{\partial \xi}\right)^2 - \kappa_0\left(\mathrm{We} - \frac{1}{3}\right)\frac{\partial^3 F_0}{\partial \xi^3}\right\} + \frac{\nu_0^2}{\kappa_0}\frac{\partial^2 F_0}{\partial y^2} = 0. \tag{6}$$

Here we do not investigate the various properties of this classical KP equation (6). For a study of KP equations in the description of water waves see: Levi [41] and the book of Infeld and Rowlands [26].

In addition:

$$\zeta_0 = \frac{\partial F_0}{\partial \xi}, \tag{7}$$

but, for the determination of $\zeta_1(\xi, \tau, y)$, in the expansion (3), it is necessary first to determine the function $F_1(\xi, \tau, y)$, since

$$\zeta_1 = \frac{\partial F_1}{\partial \xi} - A(F_0), \tag{8}$$

with

$$A(F_0) = \frac{1}{2}\left(\frac{\partial F_0}{\partial \xi}\right)^2 - \kappa_0\left(\mathrm{We} - \frac{1}{2}\right)\frac{\partial^3 F_0}{\partial \xi^3} + \frac{\partial F_0}{\partial \tau}. \tag{9}$$

Now, if again F_2 and ζ_2 are assumed to be dependent on ξ (and τ, y), we may obtain from (4c) the following result:

$$\frac{\partial F_2}{\partial \xi} = \zeta_2 + \frac{\partial F_0}{\partial \xi}\frac{\partial F_1}{\partial \xi} - \kappa_0\left(\mathrm{We} - \frac{1}{2}\right)\frac{\partial^3 F_1}{\partial \xi^3} + \frac{\partial F_1}{\partial \tau} + B(F_0) - \kappa_0 \,\mathrm{We}\frac{\partial^2}{\partial \xi^2}[A(F_0)]; \tag{10a}$$

$$\begin{aligned} \frac{\partial \zeta_2}{\partial \xi} = {} & \frac{\partial^2 F_2}{\partial \xi^2} - \frac{\kappa_0}{6}\frac{\partial^4 F_1}{\partial \xi^4} + \frac{\nu_0^2}{\kappa_0}\frac{\partial^2 F_1}{\partial y^2} + 2\frac{\partial}{\partial \xi}\left(\frac{\partial F_0}{\partial \xi}\frac{\partial F_1}{\partial \xi}\right) + \frac{\partial^2 F_1}{\partial \tau\,\partial \xi} \\ & + C(F_0) - \frac{\partial}{\partial \tau}[A(F_0)] - \frac{\partial}{\partial \xi}\left[A_0(F_0)\frac{\partial F_0}{\partial \xi}\right], \end{aligned} \tag{10b}$$

with

$$B(F_0) = -\frac{\kappa_0^2}{24}\frac{\partial^5 F_0}{\partial \xi^5} + \frac{\kappa_0}{2}\left(\frac{\partial^2 F_0}{\partial \xi^2}\right)^2 - \frac{\kappa_0}{2}\frac{\partial F_0}{\partial \xi}\frac{\partial^3 F_0}{\partial \xi^3} + \kappa_0 \frac{\partial}{\partial \xi}\left[\frac{\partial F_0}{\partial \xi}\frac{\partial^2 F_0}{\partial \xi^2} - \frac{1}{2}\frac{\partial^2 F_0}{\partial \tau\,\partial \xi}\right]$$
$$+ \nu_0^2\left[\frac{1}{2\kappa_0}\left(\frac{\partial F_0}{\partial y}\right)^2 + \left(\frac{1}{2} - \mathrm{We}\right)\frac{\partial^3 F_0}{\partial \xi\,\partial y^2}\right] \tag{11}$$

and

$$C(F_0) = \frac{\kappa_0^2}{120}\frac{\partial^6 F_0}{\partial \xi^6} - \frac{\kappa_0}{6}\frac{\partial}{\partial \xi}\left[\frac{\partial F_0}{\partial \xi}\frac{\partial^3 F_0}{\partial \xi^3}\right] + \nu_0^2\left[\frac{\partial}{\partial y}\left(\frac{\partial F_0}{\partial \xi}\frac{\partial F_0}{\partial y}\right) - \frac{1}{6}\frac{\partial^4 F_0}{\partial \xi^2\,\partial y^2}\right]. \tag{12}$$

Finally, from equations (10a) and (10b), by elimination of the function ζ_2, we may derive the following inhomogeneous (but linear!) equation for the function $F_1(\xi, \tau, y)$:

$$\frac{\partial}{\partial \xi}\left\{2\frac{\partial F_1}{\partial \tau} + 3\frac{\partial F_0}{\partial \xi}\frac{\partial F_1}{\partial \xi} + \kappa_0\left(\frac{1}{2} - \mathrm{We}\right)\frac{\partial^3 F_1}{\partial \xi^3}\right\} + \frac{\nu_0^2}{\kappa_0}\frac{\partial^2 F_1}{\partial y^2}$$
$$= \frac{\partial}{\partial \xi}\left[A(F_0)\frac{\partial F_0}{\partial \xi} - B(F_0)\right] + \left(\frac{\partial}{\partial \tau} + \kappa_0\,\mathrm{We}\frac{\partial^2}{\partial \xi^2}\right)A(F_0) - C(F_0). \tag{13}$$

Thus, the *KP equation* (6) *and the linear inhomogeneous equation* (13) *describe the second-order 'KP' approximation.*

It is now well established that the KP equation is the lowest-order non-trivial consequence of a perturbation approximation of Q1DGB-equations, for the weakly dispersive case. Attempts to advance the perturbation analysis to a higher-order have been made mainly (but, for the KdV theory only!) on the basis of a reductive perturbation theory. Ichikawa and his collaborators ([46] and [47]) examined corrections to the KdV soliton in the next higher order of approximation and introduced the notion of '*the dressed KdV soliton*', that is KdV solitons involving higher-order corrections. However, in their results, which were based on the reductive perturbation method, the dressed KdV soliton involved the appearance of secular terms. These secularities have been eliminated by Sugimo and Kakutani [48] by introducing multiple space and time variables (see for example, the book by Jeffrey and Kawahara ([49], Section 7.2)).

Here, it is important to note that the KP equation admits also solitary wave solutions. Indeed, the KP equation (6), for $F_0(\xi, \tau, y)$, can be written as an equation for: $\zeta_0 = \partial F_0/\partial \xi$,

$$\frac{\partial}{\partial \xi}\left[\frac{\partial \zeta_0}{\partial \tau} + \frac{3}{2}\zeta_0\frac{\partial \zeta_0}{\partial \xi} + \frac{\kappa_0}{6}\frac{\partial^3 \zeta_0}{\partial \xi^3}\right] + \frac{\nu_0^2}{2\kappa_0}\frac{\partial^2 \zeta_0}{\partial y^2} = 0\,, \tag{14}$$

when $\mathrm{We} \equiv 0$. Now, we can seek as a solution of (14):

$$\zeta_0 = \zeta_0(\theta)\,, \quad \text{with } \theta = \xi - p\tau + qy\,. \tag{15}$$

In this case we obtain a 'KP soliton' solution of (14) in the following (dimensionless) form:

$$\zeta_0 = \operatorname{sech}^2\left[\xi - \left(1 + \frac{\nu_0^4}{3}\right)\tau + \nu_0^2 y\right], \tag{16}$$

with

$$\kappa_0 = \tfrac{3}{4}, \qquad p = 1 + q\nu_0^2/3, \qquad q \equiv \nu_0^2, \tag{17}$$

and where ν_0 is a parameter describing the (small!) inclination of the wave to the main direction of propagation. In the absence of the y direction (where $\nu \equiv 0$; one-dimensional case), the solution (16) reduces to the KdV soliton solution.

Introducing solution (16) (for $\partial F_0/\partial\xi = \zeta_0$) into (13), it is possible to find the second-order term $F_1(\theta)$, as a function of θ and in this case, by analogy it is also possible to introduce the notion of the '*dressed KP soliton*', that is the KP soliton involving second-order corrections. However, this dressed KP soliton solution can also involve the appearance of secular terms, and for the elimination of these secularities, in addition to ξ and τ, in our above reductive perturbation method, the following new slow variables must now be introduced:

$$X = \varepsilon(x - t) \equiv \varepsilon\xi\,; \qquad T = \varepsilon^2 t \equiv \varepsilon\tau\,. \tag{18}$$

Naturally, in this case equation (13) for F_1 changes and in the transformed equation for $F_1(\xi, \tau, y, X, T)$ some new terms appear, with derivatives with respect to X and T. Consequently, it is possible to assume as a soliton solution the following:

$$F_1(\theta; X, T) = \chi(X, T)\,\mathrm{sech}^2\{\lambda(X, T)[\theta + \varphi(X, T)]\}\,. \tag{19}$$

Now, we may use the added freedom for the elimination of secularity-producing terms. But, we will not go into details of such generalizations here.

5. Evolution equations for slowly modulated weakly non-linear water waves

5.1. A generalized KP equation

There do exist other non-linear model equations, having a nature similar to the Boussinesq, KdV and KP equations, which arise naturally in the asymptotic theory of non-linear water waves.

As an important example we consider in Section 5.2 below, the derivation of the so-called 'non-linear Schrödinger–Poisson' system of two equations in the long-wave limit (Freeman–Davey NLSP equations).

It is now known that for waves on the waver surface, the modulation of a wave packet in two space dimensions due to dispersion and weak non-linearity is described, to the leading order of approximation, by the NLSP coupled system of two equations for the wave amplitude and velocity potential.

For the derivation (see Section 5.2) of this system it is necessary to assume, first, that the *carrier wave* (of the form of $\exp(ikp)$) propagates at the phase velocity C_p,

$$p = x - C_p t\,, \tag{1}$$

but the amplitude modulation travels at the corresponding group velocity C_g,

$$q = (1/\kappa_0)[x - C_g t]\,, \tag{2}$$

where C_p and C_g are given (in dimensionless form and according to linearized theory) by:

$$C_p = 1 - (1/6)\delta^2 + \cdots \quad \text{and} \quad C_g = 1 - (1/2)\delta^2 + \cdots, \quad \text{when } \delta \to 0\,. \tag{3}$$

Then, instead of the time t it is necessary to introduce a new slow time:

$$\tau = (\delta^2/\kappa_0^2)t\,. \tag{4}$$

But before deriving the NLSP equations, it is necessary, however, to modify the form of Q1DGB equations (3.8a,b), and from this new form it will then be possible to derive a so-called '*generalized KP*' equation.

Therefore, let us return to the Q1DGB equations (3.8a,b) and assume, again, that:

$$\hat{T} = \kappa_0 \,\mathrm{We}\, \varepsilon\,, \quad \text{with We} = O(1)\,,$$

let us also take, in equations (3.8a,b), instead of ε, the ratio: δ^2/κ_0. In this case, instead of (3.8a,b) we find two equations for:

$$F(t, x, \eta; \delta^2, \kappa_0) \quad \text{and} \quad \zeta(t, x, \eta; \delta^2, \kappa_0)\,, \quad \text{where } \eta = y/\nu_0\,. \tag{5}$$

Next, according to (1)–(4), it is necessary to introduce in these new equations, for F and ζ, the following new independent variables:

$$x_1 = \frac{x}{\kappa_0}, \quad t_1 = \frac{\delta^2}{6}t, \quad t_2 = \frac{t}{\kappa_0}, \quad t_3 = \frac{\delta^2}{2\kappa_0}t \quad \text{and} \quad \tau = \frac{\delta^2}{\kappa_0^2}t\,, \tag{6}$$

and also

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x} + \frac{1}{\kappa_0}\frac{\partial}{\partial x_1}\,, \tag{7a}$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t} + \frac{\delta^2}{6}\frac{\partial}{\partial t_1} + \frac{1}{\kappa_0}\frac{\partial}{\partial t_2} + \frac{\delta^2}{2\kappa_0}\frac{\partial}{\partial t_3} + \frac{\delta^2}{\kappa_0^2}\frac{\partial}{\partial \tau}\,, \tag{7b}$$

for the derivatives.

In this case, for the functions:

$$F(x, x_1, t, t_1, t_2, t_3, \tau, \eta; \delta^2, \kappa_0) \quad \text{and} \quad \zeta(x, x_1, t, t_1, t_2, t_3, \tau, \eta; \delta^2, \kappa_0)\,,$$

we again find two equations. Now, the dependent variables (the functions F and ζ) in these last two equations are expanded in the following form:

$$\begin{aligned} F &= F_0(p, q, \eta, \tau; \kappa_0) + \frac{\delta^2}{\kappa_0}F_1 + \cdots\,, \\ \zeta &= \zeta_0(p, q, \eta, \tau; \kappa_0) + \frac{\delta^2}{\kappa_0}\zeta_1 + \cdots\,, \end{aligned} \tag{8}$$

with $p = x - t + t_1$ and $q = x_1 - t_2 + t_3$.

We note that when, later, $\alpha = 1/\kappa_0$, is taken to be small, then the independent variables q, η and τ will be *slow variables modulating the rapid variation characterized by the variable p*.

Substituting (8), with (1)–(4), in the two equations for F and ζ, and equating terms of order δ^2, with α fixed gives, to order δ^2,

$$\zeta_0 = \frac{\partial F_0}{\partial X}, \tag{9}$$

and

$$\alpha\zeta_1 - \alpha\frac{\partial F_1}{\partial X} = -\alpha^2\frac{\partial F_0}{\partial \tau} - \frac{1}{6}\frac{\partial F_0}{\partial p} - \frac{1}{2}\alpha\frac{\partial F_0}{\partial q} - \frac{1}{2}\frac{\partial^3 F_0}{\partial X^3} - \frac{1}{2}\alpha\left(\frac{\partial F_0}{\partial X}\right)^2 - \mathrm{We}\frac{\partial^2 \zeta_0}{\partial X^2}; \tag{10}$$

and

$$\alpha\frac{\partial \zeta_1}{\partial X} - \alpha\frac{\partial^2 F_1}{\partial X^2} = \alpha^2\frac{\partial \zeta_0}{\partial \tau} + \frac{1}{6}\frac{\partial \zeta_0}{\partial p} + \frac{1}{2}\alpha\frac{\partial \zeta_0}{\partial q} + \alpha\frac{\partial F_0}{\partial X}\frac{\partial \zeta_0}{\partial X} + \alpha\zeta_0\frac{\partial^2 F_0}{\partial X^2} - \frac{1}{6}\frac{\partial^4 F_0}{\partial X^4} + \alpha^2\frac{\partial^2 F_0}{\partial \eta^2}, \tag{11}$$

with

$$\frac{\partial}{\partial X} \equiv \frac{\partial}{\partial p} + \alpha\frac{\partial}{\partial q}.$$

As expected, the first equation (9) is insufficient to determine both functions ζ_0 and F_0! Because of this, it is necessary to consider equations (10) and (11) (second order in δ^2) to obtain a consistency condition to do this. To obtain this consistency condition, it is sufficient to differentiate (10) by $\partial/\partial X$ and subtract from (11). As a consequence of this, the following *'generalized' KP equation* is derived:

$$2\alpha^2\frac{\partial \zeta_0}{\partial \tau} + \frac{1}{3}\frac{\partial \zeta_0}{\partial p} + \alpha\frac{\partial \zeta_0}{\partial q} + \alpha^2\frac{\partial^2 F_0}{\partial \eta^2} + 3\alpha\zeta_0\left(\frac{\partial \zeta_0}{\partial p} + \alpha\frac{\partial \zeta_0}{\partial q}\right) + \left(\frac{1}{3} - \mathrm{We}\right)\frac{\partial^3 \zeta_0}{\partial X^3} = 0. \tag{12}$$

With (9) the above GKP equation (12) suffices to determine ζ_0 and F_0, given the appropriate boundary conditions. An analogous equation has been obtained by Freeman and Davey [2], when $\mathrm{We} = 0$. Here we write this GKP equation (12), in the following detailed form:

$$2\alpha^2\frac{\partial \zeta_0}{\partial \tau} + \frac{1}{3}\frac{\partial \zeta_0}{\partial p} + \alpha\frac{\partial \zeta_0}{\partial q} + 3\alpha\zeta_0\frac{\partial \zeta_0}{\partial p} + 3\alpha^2\zeta_0\frac{\partial \zeta_0}{\partial q}$$
$$+ \frac{1}{3k^2}\left\{\frac{\partial^3 \zeta_0}{\partial p^3} + 3\alpha\frac{\partial^3 \zeta_0}{\partial p^2\,\partial q} + 3\alpha^2\frac{\partial^3 \zeta_0}{\partial p\,\partial q^2} + \alpha^3\frac{\partial^3 \zeta_0}{\partial q^3}\right\} + \alpha^2\frac{\partial^2 F_0}{\partial \eta^2} = 0, \tag{13a}$$

with

$$\zeta_0 = \frac{\partial F_0}{\partial p} + \alpha\frac{\partial F_0}{\partial q}, \tag{13b}$$

where $k^2 = [1 - 3\,\mathrm{We}]^{-1}$ and we assume that $k^2 > 0$.

5.2. Asymptotic derivation of the NLSP system of two equations in the long wave limit

From the GKP equation (13a), with (13b), we can now proceed to the direct derivation of the NLSP system of two equations, when $\alpha \ll 1$.

First, expand the unknown functions ζ_0 and F_0 in the form, with respect to α:

$$\zeta_0 = f_0 + \alpha f_1 + \alpha^2 f_2 + \cdots, \qquad F_0 = g_0 + \alpha g_1 + \alpha^2 g_2 + \cdots. \tag{14}$$

In this case for f_0, g_0, f_1, g_1, f_2 and g_2, we obtain the following equations, if we, successively, equate like terms in α^0, α^1 and α^2:

$$f_0 = \frac{\partial g_0}{\partial p}; \qquad f_n = \frac{\partial g_n}{\partial p} + \frac{\partial g_{n-1}}{\partial q}, \quad \text{for } n = 1 \text{ and } 2, \tag{15}$$

and

$$L(f_0) \equiv \frac{\partial^2 f_0}{\partial p^2} + k^2 f_0 = 0; \tag{16a}$$

$$L\left(\frac{\partial f_1}{\partial p}\right) = -3k^2\left\{\frac{\partial f_0}{\partial q} + 3f_0\frac{\partial f_0}{\partial q} + \frac{1}{k^2}\frac{\partial^3 f_0}{\partial p^2\,\partial q}\right\}; \tag{16b}$$

$$L\left(\frac{\partial f_2}{\partial p}\right) = -3k^2\left\{2\frac{\partial f_0}{\partial \tau} + 3f_0\frac{\partial f_0}{\partial q} + \frac{\partial^2 g_0}{\partial \eta^2} + \frac{\partial f_1}{\partial q} + 3\frac{\partial}{\partial p}(f_0 f_1) + \frac{1}{k^2}\frac{\partial^2}{\partial p\,\partial q}\left(\frac{\partial f_1}{\partial p} + \frac{\partial f_0}{\partial q}\right)\right\}. \tag{16c}$$

From (16a) and (15), for f_0, we determine f_0 and g_0 in the following form:

$$f_0 = A_{01}E + A_{01}^* E^{-1}; \tag{17a}$$

$$g_0 = B_{00} + B_{01}E + B_{01}^* E^{-1}, \tag{17b}$$

where $E = \exp(ikp)$ and '*' denotes the complex conjugate.
In the above relations (17):

$$B_{01} = -\frac{i}{k}A_{01}; \qquad B_{01}^* = \frac{i}{k}A_{01}^*. \tag{18}$$

Next, from (15), for f_1, we determine the function f_1 in the following form:

$$f_1 = A_{10} + A_{11}E + A_{11}^* E^{-1} + \frac{\partial g_1}{\partial p}, \tag{19}$$

with

$$A_{10} = \frac{\partial B_{00}}{\partial q}, \qquad A_{11} = \frac{\partial B_{01}}{\partial q}, \qquad A_{11}^* = \frac{\partial B_{01}^*}{\partial q}. \tag{20}$$

Now, from (16b) we find for the function g_1 the following equation, if we take into account the expression of f_0, according to (17a) and also (19),

$$L\left(\frac{\partial^2 g_1}{\partial p^2}\right) = -\frac{9}{2}k^2[A_{01}^2 E^2 + (A_{01}^*)^2 E^{-2}], \tag{21}$$

and the expression of g_1 is:

$$g_1 = B_{10} + B_{12}E^2 + B_{12}^*E^{-2}, \tag{22}$$

with

$$B_{12} = -\frac{3i}{4k}A_{01}^2, \qquad B_{12}^* = \frac{3i}{4k}(A_{01}^*)^2. \tag{23}$$

We specify that in the above relations (17)–(20), (22) and (23) the coefficients:

$$A_{01}, A_{01}^*, B_{00}, B_{01}, B_{01}^*, A_{10}, A_{11}, A_{11}^*, B_{10}, B_{12}, B_{12}^*,$$

are all functions of q, τ and η.

Now, it is necessary to consider equations (15), for f_2, and (16c). In fact, equation (16c) is an equation for f_2:

$$L\left(\frac{\partial f_2}{\partial p}\right) = -3k^2\left\{N_0 + \sum_{n=1}^{3}(N_nE^n + N_n^*E^{-n})\right\}, \tag{24}$$

where

$$N_0 = \frac{\partial A_{10}}{\partial q} + 3\left(A_{01}\frac{\partial A_{01}^*}{\partial q} + A_{01}^*\frac{\partial A_{01}}{\partial q}\right) + \frac{\partial^2 B_{00}}{\partial \eta^2}; \tag{25}$$

$$N_1 = 2\frac{\partial A_{01}}{\partial \tau} + \frac{i}{k}\frac{\partial^2 A_{01}}{\partial q^2} + 3ik(A_{01}A_{10} + A_{01}^*A_{12}) + \frac{\partial^2 B_{01}}{\partial \eta^2}; \tag{26}$$

$$N_2 = \frac{3}{2}\frac{\partial A_{01}^2}{\partial q} + 6ikA_{01}A_{11} + 3A_{01}\frac{\partial A_{01}}{\partial q} - 4\frac{\partial A_{12}}{\partial q} \equiv 0; \tag{27}$$

$$N_3 = 9ikA_{01}A_{12}, \tag{28}$$

when the above expressions (17a) and (19) for f_0 and f_1 are utilized. But from (15), for f_2, and (22):

$$f_2 = A_{20} + A_{22}E^2 + A_{22}^*E^{-2} + \frac{\partial g_2}{\partial p}, \tag{29}$$

with

$$A_{20} = \frac{\partial B_{10}}{\partial q}, \qquad A_{22} = \frac{\partial B_{12}}{\partial q}, \qquad A_{22}^* = \frac{\partial B_{12}^*}{\partial q}, \tag{30}$$

and consequently, for the left-hand side of equation (24), the following relation is derived

$$L\left(\frac{\partial f_2}{\partial p}\right) = \frac{\partial}{\partial p}L\left(\frac{\partial g_2}{\partial p}\right) - 6ik^3[A_{22}E^2 - A_{22}^*E^{-2}]. \tag{31}$$

Obviously, from (29), (31) and (24), for g_2 it is necessary to resolve the following equation

$$\frac{\partial}{\partial p}L\left(\frac{\partial g_2}{\partial p}\right) = -3k^2\{-2ikA_{22}E^2 + 2ikA_{22}^*E^{-2} + N_3E^3 + N_3^*E^{-3}\}, \tag{32}$$

and consequently we find the following *two compatibility conditions*:

$$N_0 = 0\,, \qquad N_1 = 0\,. \tag{33}$$

Finally, we obtain for the following six functions:

$$A_{10}, A_{01}, B_{00}, B_{01}, A_{11} \quad \text{and} \quad A_{12} \equiv 2ikB_{12}\,,$$

the following four relations

$$B_{01} = -\frac{i}{k}A_{01}\,; \qquad \frac{\partial B_{00}}{\partial q} = A_{10}\,; \qquad A_{11} = \frac{\partial B_{01}}{\partial q}\,; \qquad B_{12} = -\frac{3i}{4k}A_{01}^2\,, \tag{34}$$

and two equations (from (33), (25) and (26)):

$$\frac{\partial A_{10}}{\partial q} + 3\frac{\partial}{\partial q}|A_{01}|^2 + \frac{\partial^2 B_{00}}{\partial q^2} = 0 \tag{35}$$

and

$$2\frac{\partial A_{01}}{\partial \tau} + \frac{i}{k}\frac{\partial^2 A_{11}}{\partial q^2} + 3ik(A_{01}A_{10} + A_{01}^* A_{12}) + \frac{\partial^2 B_{01}}{\partial \eta^2} = 0\,, \tag{36}$$

since $A_{01}^* A_{01} \equiv |A_{01}|^2$.

From (34)–(36) the functions A_{10}, A_{11}, B_{01}, A_{12} are eliminated and we find for A_{01} and B_{00} the following *NLSP system* of two equations:

$$\frac{\partial^2 B_{00}}{\partial q^2} + \frac{\partial^2 B_{00}}{\partial \eta^2} + 3\frac{\partial}{\partial q}|A_{01}|^2 = 0\,; \tag{37}$$

and

$$2ik\frac{\partial A_{01}}{\partial \tau} - \frac{\partial^2 A_{01}}{\partial q^2} + \frac{\partial^2 A_{01}}{\partial \eta^2} - \frac{9}{2}k^2 A_{01}|A_{01}|^2 = 3k^2 A_{01}\frac{\partial B_{00}}{\partial q}\,. \tag{38}$$

From equations (37) and (38) we may rederive the non-linear Schrödinger and Poisson equations obtained by Freeman and Davey [2] (but only for $k \equiv 1$). Hence, the *amplitude modulation* A_{01} of a progressive wave-packet of small amplitude (progressing in quasi-one direction on water of finite depth) may be described by a NLS equation (38), coupled to a Poisson (P) type equation (37) for the *mean part of the velocity potential of the flow* B_{00}. Note that the equations of non-linear Schrödinger type in $1 + 1$ and $2 + 1$ dimensions, have been obtained from integrable PDE's by Calogero and Maccari [7].

We note also that the NLSP system of two equations (37), (38), was derived by Davey and Stewartson [32], when $k = 1$, in the long-wave limit ($\varepsilon \to 0$ and then $\delta \to 0$), but without any formal justification. According to Davey and Stewartson [32], in fact, the double limit δ, $\alpha \to 0$ is uniform, since the order in which the limits are taken is immaterial. Equations (37) and (38) suffice to determine A_{01} and B_{00}, given appropriate boundary conditions. On physical grounds a 'reasonable' boundary condition is that, for any fixed time τ, the wave completely dies away sufficiently far from its centre so that:

$$|A_{01}| \to 0 \quad \text{and} \quad \frac{\partial B_{00}}{\partial q} \to 0, \quad \frac{\partial B_{00}}{\partial \eta} \to 0 \quad \text{as } |q^2 + \eta^2| \to \infty . \tag{39}$$

Furthermore, if we suppose that at time $t = 0$ a progressive wave is established such that the elevation of the free surface is raised to $z = \varepsilon\zeta^0(\alpha x, \eta)\exp(ix) + \text{C.C.}$ (in dimensionless form, according to (2.5), and for $k = 1$), then the appropriate initial condition for A_{01} is:

$$A_{01}(q, \eta, 0) = \zeta^0(\alpha x, \eta)\,; \qquad q|_{t=0} \equiv \alpha x . \tag{40}$$

Thus, at this stage, we may associate the following two *explicit* asymptotic expansions for the functions ζ_0 and F_0:

$$\zeta_0 = A_{01}E + A_{01}^* E^{-1} + \alpha\left[\frac{\partial B_{00}}{\partial q} - \frac{i}{k}\frac{\partial A_{01}}{\partial q}E + \frac{i}{k}\frac{\partial A_{01}^*}{\partial q}E^{-1} + \frac{3}{2}A_{01}^2E^2 + \frac{3}{2}(A_{01}^*)^2E^{-2}\right] + \cdots , \tag{41}$$

and

$$F_0 = B_{00} - \frac{i}{k}A_{01}E + \frac{i}{k}A_{01}^*E^{-1} , \tag{42}$$

where $E = \exp(ikp)$.

But, now according to our above rational asymptotic derivation, in principle we can extend these asymptotic expansions (41) and (42) up to the term of the order of α^3, for ζ_0, and up to the term of the order of α^2, for F_0. For this it is necessary to resolve, first, equation (32) for g_2.

Surprisingly, the expression of g_2 is then of the following form:

$$g_2 = B_{20} + \frac{3}{8k^2}\frac{\partial A_{01}^2}{\partial q}E^2 + \frac{3}{8k^2}\frac{\partial (A_{01}^*)^2}{\partial q}E^{-2} - \frac{9}{16}iA_{01}^3E^3 + \frac{9}{16}i(A_{01}^*)^3E^{-3} , \tag{43}$$

and from (29) we obtain for f_2 the following expression (the terms with E^2 and E^{-2} cancel out!):

$$f_2 = \frac{\partial B_{10}}{\partial q} + \frac{27}{16}A_{01}^3E^3 + \frac{27}{16}(A_{01}^*)^3E^{-3} . \tag{44}$$

Similarly, for the function g_1, from the relations (22) and (23), the following representation is obtained:

$$g_1 = B_{10} - \frac{3i}{4k}A_{01}^2E^2 + \frac{3i}{4k}(A_{01}^*)^2E^{-2} . \tag{45}$$

Therefore, if we should extend (41) and (42) then it is necessary to determine the function $B_{10}(q, \tau, \eta)$, and for this we can consider the equation for the function f_3 and determine the structure of the right-hand side of this equation!

From the GKP equation (13a) we deduce easily the following equation for f_3:

$$L\left(\frac{\partial f_3}{\partial p}\right) = -3k^2\left\{M_0 + \sum_{n=1}^{4} M_n E^n + M_n^* E^{-n}\right\}, \tag{46}$$

if we take into account the expressions: (17a), for f_0, (19), for f_1 and (44), for f_2, where

$$M_0 = 2\frac{\partial A_{10}}{\partial \tau} + \frac{\partial A_{20}}{\partial q} + 3A_{01}\frac{\partial A_{11}^*}{\partial q} + 3A_{01}^*\frac{\partial A_{11}}{\partial q} + 3A_{11}^*\frac{\partial A_{01}}{\partial q} + 3A_{11}\frac{\partial A_{01}^*}{\partial q} + \frac{\partial^2 B_{10}}{\partial \eta^2}. \tag{47}$$

We do not write the corresponding expressions for the terms proportional to E, E^2, E^3 and E^4, since we only intend to derive an equation for B_{10}. From (47) the equation for B_{10} is of the following form:

$$\frac{\partial^2 B_{10}}{\partial q^2} + \frac{\partial^2 B_{10}}{\partial \eta^2} = -2\frac{\partial^2 B_{00}}{\partial \tau\, \partial q} + \frac{3i}{k}\left[A_{01}^*\frac{\partial^2 A_{01}}{\partial q^2} - A_{01}\frac{\partial^2 A_{01}^*}{\partial q^2}\right]. \tag{48}$$

Finally, we may now perform the following consistent and significant expansions for ζ_0 and F_0:

$$\zeta_0 = A_{01}E + A_{01}^*E^{-1} + \alpha\left\{\frac{\partial B_{00}}{\partial q} - \frac{i}{k}\frac{\partial A_{01}}{\partial q}E + \frac{i}{k}\frac{\partial A_{01}^*}{\partial q}E^{-1} + \frac{3}{2}A_{01}^2E^2 + \frac{3}{2}(A_{01}^*)^2E^{-2}\right\}$$
$$+ \alpha^2\left\{\frac{\partial B_{10}}{\partial q} + \frac{27}{16}A_{01}^3E^3 + \frac{27}{16}(A_{01}^*)^3E^{-3}\right\} + O(\alpha^3); \tag{49}$$

and

$$F_0 = B_{00} - \frac{i}{k}A_{01}E + \frac{i}{k}A_{01}^*E^{-1} + \alpha\left\{B_{10} - \frac{3}{4k}iA_{01}^2E^2 + \frac{3}{4k}i(A_{01}^*)^2E^{-2}\right\} + O(\alpha^2). \tag{50}$$

The relevant equations for the functions $A_{01}(\tau, q, \eta)$, $B_{00}(\tau, q, \eta)$ and $B_{10}(\tau, q, \eta)$ are equations (37), (38) and (48).

It should be observed that we can in principle extend the above expansions (49) and (50), if we consider the corresponding equations for $f_4, f_5, \ldots$ and $g_4, g_5, \ldots$. But, in this case it is necessary also to introduce, with the slow variables τ, q and η, a similar number of new slow variables as, for example,

$$\tau_1 = \alpha\tau, \qquad q_1 = \alpha q, \ldots. \tag{51}$$

Indeed, at the right-hand side of equation (46) the term proportional to E is of the following form:

$$M_1 = 2\frac{\partial A_{11}}{\partial \tau} + 2ik[A_{20}A_{01} + A_{10}A_{11} + A_{11}^*A_{12}]$$
$$+ 3\frac{\partial}{\partial q}[A_{01}A_{10} + A_{01}^*A_{12}] + \frac{i}{k}\frac{\partial^2 A_{11}}{\partial q^2} + \frac{1}{3k^2}\frac{\partial^3 A_{01}}{\partial q^3}, \tag{52}$$

and this gives a new compatibility relation, which is in fact a new equation between A_{01}, B_{00} and B_{10}, which satisfies equations (37), (38) and (48)! It is not evident that this last new equation is an identity. Therefore, seemingly the problem for A_{01}, B_{00} and B_{10} is *overdetermined*! To remedy this difficulty we can assume that our functions A_{01}, B_{00} and B_{10}

are also dependent of the slow variables, τ_1, $q_1, \ldots$, according to (51)! Clearly more research is needed in this direction!

In conclusion, it is possible to assert therefore that the double limit in which, first, $\delta \to 0$ and then $\alpha \to 0$, described in this section, is valid and correct. Since the more formal procedure in which first $\kappa_0 \to \infty$ and then $\delta \to 0$ (note that $\delta^2 = \varepsilon/\alpha$) yields the same result, the double limit δ and $1/\kappa_0 = \alpha \to 0$ must be valid and uniform with equations (37), (38) and (48) as appropriate evolutionary equations.

These evolutionary equations (37), (38) and (48) are only relevant equations for the long water-wave limit in shallow water when:

$$\varepsilon \to 0 \quad \text{and} \quad \delta \to 0 .$$

As this is remarked upon in the Freeman and Davey [2] paper, the two evolutionary equations (37), (38) and the new equation (48) are derived on he basis of a double-expansion procedure which assumed that an expansion in terms of δ could be used first, followed by an expansion in α. Such a procedure would seem to imply that the parameters δ and α were quite independent of each other. A close examination of the method indicates, however, that the results still remain true, even if α is dependent on δ. At first sight, the retention of terms of order $\alpha\delta^2$ in deriving equations (10) and (11) with the neglect of terms of order δ^4 in (3.7) would suggest that some restriction on the size of α relative to δ was implied. However, it should be realized that the terms of order δ^4 neglected in (3.7) are just those terms which are zero to first order in α because of the value of $C_p = 1 - \delta^2/6 + \cdots$ taken in accordance with linearized theory, to achieve exactly that similar observation applies to certain terms of order $\alpha\delta^4$, because of the choice of $C_g = 1 - \delta^2/2 + \cdots$.

5.3. Matching between KP and Davey–Stewartson GNLSP equations

Now, it is also important to note that, in a more general case, it is possible to derive a coupled system of evolution equations for the packet of water waves directly from the dimensionless problem: (2.11), (2.13)–(2.15) or (2.16), when $\varepsilon \to 0$ and $\Delta = \nu_0 \varepsilon$, but with δ and ν_0 fixed!

For the derivation of this system of two equations (two-dimensional general non-linear Schrödinger equation coupled with a Poisson equation), see the papers by: Benney and Roskes [3], Davey and Stewartson [32], Djordjevic and Redekopp [33], Ablowitz and Segur [34] and also the book by Mei ([12], pp. 607–618).

Without surface tension (when $\mathrm{We} \equiv 0$) and with dimensionless variables, our starting problem is the classical water-wave problem: Laplace's equation (2.11), within the water, together with the surface conditions (2.14), (2.15) and the flat-bottom boundary condition (2.13).

Only a brief outline of the perturbation analysis need to be given here.

The wavelength of the carrier wave is taken to be $O(1)$ as $\varepsilon \to 0$, and this corresponds to δ *being fixed* in the limit process:

$$\varepsilon \to 0 \quad \text{and} \quad \Delta = \nu_0 \varepsilon \, , \quad \text{with } \nu_0 \text{ and } \delta \text{ fixed} . \tag{53}$$

Indeed, as shown by the earlier work of Benney and Newell [50] and Davey and Stewartson [32], it proves convenient to introduce the following multiple slow scales:

$$\hat{q} = \varepsilon(x - C_g t)\,, \qquad \hat{\eta} = \varepsilon \frac{\delta}{\Delta} y \equiv \frac{\delta}{\nu_0} y = \delta\eta\,, \qquad \hat{\tau} = \varepsilon^2 t\,, \tag{54}$$

and the carrier wave moves at the phase speed C_p and the amplitude modulation moves at the corresponding group speed C_g, although the specific forms of C_p and C_g are not assumed a priori.

The wavetrain is so constructed that it be periodic (to all orders in ε) in $\hat{p} = x - C_p t$, with fundamental periodicity $\hat{E} = \exp(i\hat{p})$ and amplitude modulation described by the scaled coordinates (54), whence higher-order terms (in the series expansions in ε, according to (55)) must contain higher harmonics generated by the non-linear coupling.

Now, if as a solution to our classical water-wave problem (2.11), (2.13)–(2.15), with (54) and the periodicity in $\hat{p}$, the following asymptotic expansions are assumed:

$$\phi = \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \cdots \quad \text{and} \quad \zeta = \zeta_0 + \varepsilon\zeta_1 + \varepsilon^2\zeta_2 + \cdots\,, \tag{55}$$

then we may derive for the functions ϕ_n and ζ_n, $n = 0, 1, 2, \ldots$, the following problems:

$$\left.\begin{aligned} &\frac{\partial^2\phi_n}{\partial z^2} + \delta^2 \frac{\partial^2\phi_n}{\partial \hat{p}^2} = -F_n\,; \\ &\left.\frac{\partial\phi_n}{\partial z}\right|_{z=-1} = 0\,; \\ &\left.\frac{\partial\phi_n}{\partial z}\right|_{z=0} + \delta^2 C_p \frac{\partial\zeta_n}{\partial\hat{p}} = G_n|_{z=0}\,, \end{aligned}\right\} \tag{56}$$

and

$$\zeta_n = C_p \left.\frac{\partial\phi_n}{\partial\hat{p}}\right|_{z=0} + H_n|_{z=0}\,, \tag{57}$$

where $F_0 = 0$, $G_0 = 0$, $H_0 = 0$ and F_n, G_n and H_n ($n = 1, 2$) are known functions.

For ϕ_0 and ζ_0 the following leading solution is obtained:

$$\phi_0 = \phi_{00}(\hat{q}, \hat{\eta}, \hat{\tau}) + F_{00}(z)[A(\hat{q}, \hat{\eta}, \hat{\tau})\hat{E} + A^*\hat{E}^{-1}]\,, \tag{58a}$$

$$\zeta_0 = C_p \left.\frac{\partial\phi_0}{\partial\hat{p}}\right|_{z=0} = iC_p[A\hat{E} - A^*\hat{E}^{-1}]\,, \tag{58b}$$

with

$$F_{00}(z) = \cosh[\delta(z+1)]/\cosh(\delta)\,; \qquad \hat{E}^{-1} \equiv \exp(-i\hat{p})\,,$$

and C_p is calculated from the dispersion relation for the linear theory,

$$C_p = \frac{\omega(\delta)}{\delta}\,, \qquad \omega(\delta) = \sqrt{\sigma\delta}\,, \quad \text{with } \sigma = \tanh(\delta)\,. \tag{59}$$

Next, for ϕ_1, we have a non-homogeneous problem, and this problem is compatible if and only if:

$$C_g = C_p[\sigma + \delta(1 - \sigma^2)]/2\sigma \equiv \frac{\mathrm{d}\omega(\delta)}{\mathrm{d}\sigma}\,, \tag{60}$$

again according to linear theory.

It is possible to continue this analysis and to obtain ϕ_2 (and also ζ_2), once the solutions for the mean flow (ϕ_0 and ζ_0) and second harmonic (ϕ_1 and ζ_1) have been found. Evaluation of F_2, G_2 and H_2 is a straightforward, but rather tedious task!

Then, upon using the boundary condition on $z=0$ in (56), for $n=2$, it is possible also to find, from the expression for G_2, that the leading-order mean flow or long-wave component $\phi_{00}(\hat{q}, \hat{\tau}, \hat{\eta})$ is prescribed by the equation:

$$(1-C_g^2)\frac{\partial^2\phi_{00}}{\partial\hat{q}^2}+\frac{\partial^2\phi_{00}}{\partial\hat{\eta}_2}=-[2C_p+C_g(1-\sigma^2)]\frac{\partial}{\partial\hat{q}}|A|^2\,. \tag{61}$$

This last equation, (61), shows that the long-wave component ϕ_{00} is generated by the self-interaction of the short wave (characterized by the amplitude function $A(\hat{q}, \hat{\tau}, \hat{\eta})$).

Finally, upon comparing the first-harmonic terms in the condition on $z=0$ in (56), for $n=2$, with the corresponding G_2 and the expression for ζ_2, with the corresponding H_2, we find that the derived two equations are compatible only if the amplitude function satisfies the following non-linear (Schrödinger) evolution equation:

$$\begin{aligned}2iC_p\frac{\partial A}{\partial\hat{\tau}}-[C_g^2-(1-\sigma^2)(1-\sigma\delta)]\frac{\partial^2 A}{\partial\hat{q}^2}+C_pC_g\frac{\partial^2 A}{\partial\hat{\eta}^2}\\=[2C_p+C_g(1-\sigma^2)]A\frac{\partial\phi_{00}}{\partial\hat{q}}+\left[\frac{9}{2\sigma^2}-6+\frac{13}{2}\sigma^2-\sigma^4\right]A|A|^2\,.\end{aligned} \tag{62}$$

Davey–Stewartson *GNLSP equations* (62) and (61) together describe the evolution of the progressive wave, to first order in ε, *with* δ *fixed*.

For the *matching* between KP and this GNLSP system of two coupled equations (62), (61), we consider in (61), (62) the shallow-water limit: $\delta\to 0$. In this case, in place of (59) and (60), we have for C_p and C_g, respectively, the relations (5.3) and in place of equation (61):

$$\delta^2\frac{\partial^2\phi_{00}}{\partial\hat{q}^2}+\frac{\partial^2\phi_{00}}{\partial\hat{\eta}^2}=-3\frac{\partial|A|^2}{\partial\hat{q}}\,, \tag{63a}$$

since $\sigma=\delta-\delta^3/3+\cdots$, when $\delta\to 0$. By analogy, instead of (62) we find:

$$2i\delta^2\frac{\partial A}{\partial\hat{\tau}}-\delta^4\frac{\partial^2 A}{\partial\hat{q}^2}+\delta^2\frac{\partial^2 A}{\partial\hat{\eta}^2}=3\delta^2 A\frac{\partial\phi_{00}}{\partial\hat{q}}+\frac{9}{2}A|A|^2\,. \tag{63b}$$

Now it is necessary to compare the slow variables (54), ($\hat{q}$, $\hat{\tau}$, $\hat{\eta}$) with the GKP equation variables (2), (4), (q,τ) and $\eta=y/\nu_0$.

In this case we may deduce from this comparison the following relations:

$$\hat{q}=\delta^2 q\,,\qquad \hat{\eta}=\delta\eta \quad\text{and}\quad \hat{\tau}=\delta^2\tau\,, \tag{64}$$

and as a result rederive the NLSP system of two equations (37) and (38), but for A and ϕ_{00}:

$$\left.\begin{aligned}&\frac{\partial^2\phi_{00}}{\partial q^2}+\frac{\partial^2\phi_{00}}{\partial\eta^2}+3\frac{\partial|A|^2}{\partial q}=0\,;\\&2\mathrm{i}\frac{\partial A}{\partial\tau}-\frac{\partial^2 A}{\partial q^2}+\frac{\partial^2 A}{\partial\eta^2}-\frac{9}{2}A|A|^2-3A\frac{\partial\phi_{00}}{\partial q}=0\,,\end{aligned}\right\} \tag{65a}$$

and with $k \equiv 1$.

Therefore, it is clear that (62) and (61) match with (37), (38), when $\delta \to 0$, i.e.:

$$\phi_{00} \to B_{00} \quad \text{and} \quad A \to A_{01} \,. \tag{66}$$

Thus, the long-wave limit of the Davey–Stewartson GNLSP-equations, (61), (62), matches precisely with the short-wave limit KP equation (4.6).

Then, it is confirmed that there is a measure of agreement between the GNLSP-equations for long waves ($\delta \to 0$), and the KP equation for short waves ($\kappa_0 \to \infty$). Stated more formally:

$$\operatorname*{Lim}_{\delta \to 0} [(61), (62)] = \operatorname*{Lim}_{\kappa_0 \to \infty} [(4.6)] \,, \tag{67}$$

and, since matching occurs, the coefficients given in (61), (62), when $\delta \to 0$, can be checked against those deduced from (4.6), when $\kappa_0 \to \infty$.

For the KdV and single non-linear Schrödinger equations in the one-dimensional case and for more details of the matching procedure, see Johnson ([51] and [52], pp. 25–43).

6. Influence of an uneven bottom

6.1. Quasi-one-dimensional Boussinesq equations for a variable depth

In dimensionless form, according to (2.11), (2.14), (2.15) and (2.18), we have in this case the following problem:

$$\frac{\partial^2 \phi}{\partial z^2} + \delta^2 \frac{\partial^2 \phi}{\partial x^2} + \Delta^2 \frac{\partial^2 \phi}{\partial y^2} = 0 \,, \; -h(\hat{x}, \hat{y}, \alpha) < z < \varepsilon \zeta(x, y, t) \,; \tag{1a}$$

$$\frac{\partial \phi}{\partial z} = \delta^2 \frac{\partial \zeta}{\partial t} + \varepsilon \delta^2 \frac{\partial \phi}{\partial x} \frac{\partial \zeta}{\partial x} + \varepsilon \Delta^2 \frac{\partial \phi}{\partial y} \frac{\partial \zeta}{\partial y} \,, \quad \text{on } z = \varepsilon \zeta(x, y, t) \,; \tag{1b}$$

and

$$\frac{\partial \phi}{\partial t} + \zeta + \frac{\varepsilon}{2} \left(\frac{\partial \phi}{\partial x} \right)^2 + \frac{\varepsilon}{2} \left(\frac{\Delta}{\sigma} \right)^2 \left(\frac{\partial \phi}{\partial y} \right)^2 + \frac{\varepsilon}{2\delta^2} \left(\frac{\partial \phi}{\partial z} \right)^2 = 0 \,, \quad \text{on } z = \varepsilon \zeta(x, y, t) \,; \tag{1c}$$

$$\frac{\partial \phi}{\partial z} + \delta^2 \beta \frac{\partial \phi}{\partial x} \frac{\partial h}{\partial x} + \Delta^2 \gamma \frac{\partial \phi}{\partial y} \frac{\partial h}{\partial y} = 0 \,, \quad \text{on } z = -h(\hat{x}, \hat{y}; \alpha) \,, \tag{1d}$$

where

$$h(\hat{x}, \hat{y}; \alpha) = 1 - \alpha G(\hat{x}, \hat{y}) \,, \tag{2}$$

with

$$\hat{x} = \beta x \,; \quad \hat{y} = \gamma y \,. \tag{3}$$

According to (2.23) we assume again that

$$\delta^2 = \kappa_0 \varepsilon \quad \text{and} \quad \Delta = \nu_0 \varepsilon \,, \quad \text{when } \varepsilon \to 0 \tag{4}$$

with

$\kappa_0 = O(1)$ and $\nu_0 = O(1)$.

In his case from Laplace's equation (1a), with the uneven-bottom condition (1d), we may find the following asymptotic expansion for the velocity-potential ϕ:

$$\begin{aligned}\phi = F(t, x, y) &- \varepsilon\frac{\kappa_0}{2}(z+h)^2\frac{\partial^2 F}{\partial x^2} - \beta\varepsilon\kappa_0(z+h)\frac{\partial F}{\partial x}\frac{\partial h}{\partial \hat{x}} \\ &+ \varepsilon^2\left\{\frac{\kappa_0^2}{24}(z+h)^4\frac{\partial^4 F}{\partial x^4} - \frac{\nu_0^2}{2}(z+h)^2\frac{\partial^2 F}{\partial y^2}\right\} + \beta\varepsilon^2\frac{\kappa_0^2}{2}(z+h)^3\frac{\partial^3 F}{\partial x^3}\frac{\partial h}{\partial \hat{x}} \\ &- \gamma\varepsilon^2\nu_0^2(z+h)\frac{\partial F}{\partial y}\frac{\partial h}{\partial \hat{y}} + O(\varepsilon^3, \beta^2\varepsilon^2) .\end{aligned} \tag{5}$$

It is noticed that (5) is valid when $\gamma = O(1)$ but it is necessary that $\beta \gg \varepsilon$.

If $\beta = O(\varepsilon)$, then the fifth term in the expansion (5), proportional to $\beta\varepsilon^2$, is of order of $O(\varepsilon^3)$ and we do not take this term into account in this case! In the paper of Liu et al. [53] this last case is considered correctly and these authors have conjectured a form of the 'modified' KP equation for a variable depth (see, equation (20)). Concerning this modified KP equation, see also the paper by Xue-Nong Chen [40].

From (5) we can easily obtain the value of derivatives: $\partial\phi/\partial t$, $\partial\phi/\partial x$, $\partial\phi/\partial y$ and $\partial\phi/\partial z$ (on $z = \varepsilon\zeta(x, y, t)$) for $z = 0$.

Now, from the first of the boundary conditions (1b) on $z = \varepsilon\zeta(x, y, t)$ we arrive at the following approximate equation:

$$\begin{aligned}&\frac{\partial\zeta}{\partial t} + h\frac{\partial^2 F}{\partial x^2} + \varepsilon\left\{\frac{\partial F}{\partial x}\frac{\partial\zeta}{\partial x} + \zeta\frac{\partial^2 F}{\partial x^2} - \frac{\kappa_0}{6}h^3\frac{\partial^4 F}{\partial x^4} + \frac{\nu_0^2}{\kappa_0}h\frac{\partial^2 F}{\partial y^2}\right\} \\ &\quad = \beta\frac{\partial F}{\partial x}\frac{\partial h}{\partial \hat{x}} - \frac{3}{2}\beta\kappa_0\frac{\partial^3 F}{\partial x^3}h^2\frac{\partial h}{\partial \hat{x}} + \gamma\varepsilon\frac{\nu_0^2}{\kappa_0}\frac{\partial F}{\partial y}\frac{\partial h}{\partial \hat{y}} ,\end{aligned} \tag{6}$$

with an error of $O(\varepsilon^2)$.

Next, the second boundary condition (1c) on $z = \varepsilon\zeta(x, y, t)$ gives:

$$\frac{\partial F}{\partial t} + \zeta + \varepsilon\left\{\frac{1}{2}\left(\frac{\partial F}{\partial x}\right)^2 - \frac{\kappa_0}{2}\frac{\partial^3 F}{\partial t\,\partial x^2}h^2\right\} = \beta\varepsilon\kappa_0\frac{\partial^2 F}{\partial t\,\partial x}h\frac{\partial h}{\partial \hat{x}} , \tag{7}$$

with an error of $O(\varepsilon^2)$.

The two equations (6) and (7) are our *quasi-one-dimensional Boussinesq equations for a variable uneven bottom of the form*:

$$z = -h(\hat{x}, \hat{y}, \alpha) , \quad \text{with } \hat{x} = \beta x \quad \text{and} \quad \hat{y} = \gamma y . \tag{8}$$

If $h \equiv 1$, we rederive again, from (6) and (7), the classical Q1DB-system of two equations for F and ζ (but for We = 0):

$$\left.\begin{aligned}&\frac{\partial F}{\partial t} + \zeta + \varepsilon\left\{\frac{1}{2}\left(\frac{\partial F}{\partial x}\right)^2 + \frac{\kappa_0}{2}\frac{\partial^3 F}{\partial t\,\partial x^2}\right\} = 0 ; \\ &\frac{\partial\zeta}{\partial t} + \frac{\partial^2 F}{\partial x^2} + \varepsilon\left\{\frac{\partial}{\partial x}\left(\zeta\frac{\partial F}{\partial x}\right) - \frac{\kappa_0}{6}\frac{\partial^4 F}{\partial x^4} + \frac{\nu_0^2}{\kappa_0}\frac{\partial^2 F}{\partial y^2}\right\} = 0 .\end{aligned}\right\} \tag{9}$$

The above Boussinesq equations (6) and (7), for an uneven bottom, may also be written in the following form (with an error of $O(\varepsilon^2)$):

$$\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x}[(h + \varepsilon\zeta)u] + \varepsilon\frac{\nu_0^2}{2}\frac{\partial}{\partial y}(hv) - \varepsilon\frac{\kappa_0}{6}\frac{\partial}{\partial x}\left(h^3\frac{\partial^2 u}{\partial x^2}\right) = \beta\varepsilon\kappa_0 h^2\frac{\partial h}{\partial \hat{x}}\frac{\partial^2 u}{\partial x^2}\,; \tag{10}$$

and

$$\frac{\partial u}{\partial t} + \varepsilon u\frac{\partial u}{\partial x} + \frac{\partial \zeta}{\partial x} - \varepsilon\frac{\kappa_0}{2}h^2\frac{\partial^3 u}{\partial t\,\partial x^2} = 2\beta\varepsilon\kappa_0 h\frac{\partial h}{\partial \hat{x}}\frac{\partial^2 u}{\partial t\,\partial x}\,, \tag{11}$$

where

$$u = \frac{\partial F}{\partial x}\,, \qquad v = \frac{\partial F}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}\,. \tag{12}$$

Again from equations (6) and (7) we can eliminate the function ζ. Indeed, from (7),

$$\zeta = -\frac{\partial F}{\partial t} + \varepsilon\frac{\kappa_0}{2}\frac{\partial}{\partial x}\left(h^2\frac{\partial^2 F}{\partial x\,\partial t}\right) - \frac{\varepsilon}{2}\left(\frac{\partial F_0}{\partial x}\right)^2, \tag{13}$$

and if we take into account this above relation (13) in (6), we find for the function $F(x, y, t; h)$ a *single approximate Boussinesq equation for an uneven bottom*:

$$\begin{aligned}
&\frac{\partial^2 F}{\partial t^2} - \frac{\partial}{\partial x}\left(h\frac{\partial F}{\partial x}\right) - \varepsilon\frac{\nu_0^2}{\kappa_0}\frac{\partial}{\partial y}\left(h\frac{\partial F}{\partial y}\right) + \varepsilon\frac{\partial}{\partial t}\left[\left(\frac{\partial F}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial F}{\partial t}\right)^2\right] \\
&\quad - \varepsilon\frac{\kappa_0}{2}\frac{\partial^2}{\partial x\,\partial t}\left(h^2\frac{\partial^2 F}{\partial x\,\partial t}\right) + \varepsilon\frac{\kappa_0}{6}\frac{\partial}{\partial x}\left(h^3\frac{\partial^3 F}{\partial x^3}\right) + \beta\varepsilon\kappa_0 h^2\frac{\partial h}{\partial \hat{x}}\frac{\partial^3 F}{\partial x^3} = 0\,,
\end{aligned} \tag{14}$$

with an error of $O(\varepsilon^2)$, when $\beta \gg \varepsilon$.

Naturally, if $\beta = O(\varepsilon)$, then instead of (14) it is necessary to write the following reduced Boussinesq equation, again with an error of $O(\varepsilon^2)$ and with $\gamma = O(1)$:

$$\begin{aligned}
&\frac{\partial^2 F}{\partial t^2} - h\frac{\partial^2 F}{\partial x^2} - \varepsilon\gamma^2\frac{\nu_0^2}{\kappa_0}h\frac{\partial^2 F}{\partial \hat{y}^2} \\
&\quad + \varepsilon\left\{\frac{\partial}{\partial t}\left[\left(\frac{\partial F}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial F}{\partial t}\right)^2\right] + \frac{\kappa_0}{6}h^3\frac{\partial^4 F}{\partial x^4} - \frac{\kappa_0}{2}h^2\frac{\partial^4 F}{\partial x^2\,\partial t^2} - \gamma^2\frac{\nu_0^2}{\kappa_0}\frac{\partial h}{\partial \hat{y}}\frac{\partial F}{\partial \hat{y}}\right\} = 0\,,
\end{aligned} \tag{15}$$

for $F(x, t, \hat{y}; h)$, where $h = h(\varepsilon x, \hat{y}; \alpha)$ and $\hat{y} = \gamma y$.

6.2. *KP equation for an uneven bottom*

For the derivation of an 'extended KP' equation for an uneven bottom it is necessary to introduce in the Boussinesq equations (6) and (7), when $\beta = \varepsilon$, the following new variables:

$$\hat{x} = \varepsilon x\,, \qquad \hat{y} = \gamma y\,, \qquad \tau = \varepsilon t \tag{16a}$$

and

$$\xi = \int_0^x h^{-1/2}(\varepsilon x, \hat{y}; \alpha)\, \mathrm{d}x - t\,. \tag{16b}$$

In this case

$$\left.\begin{aligned} &\frac{\partial}{\partial x} = \varepsilon \frac{\partial}{\partial \hat{x}} + h^{-1/2}\frac{\partial}{\partial \xi}\,, \\ &\frac{\partial}{\partial y} = \gamma \frac{\partial}{\partial \hat{y}} + \mathscr{G}\frac{\partial}{\partial \xi}\,, \qquad \frac{\partial}{\partial t} = -\frac{\partial}{\partial \xi} + \varepsilon \frac{\partial}{\partial \tau}\,, \end{aligned}\right\} \tag{17a}$$

with

$$\left.\begin{aligned} &h = h(\hat{x}, \hat{y}; \alpha)\,; \\ &\mathscr{G}(\hat{x}, \hat{y}) \equiv \frac{\partial}{\partial \hat{y}} \int_0^x h^{-1/2}(\varepsilon x, \hat{y}; \alpha)\, \mathrm{d}x\,. \end{aligned}\right\} \tag{17b}$$

Now, it is assumed that

$$F = F_0(\tau, \hat{x}, \hat{y}, \xi) + \varepsilon F_1 + \cdots\,; \quad \zeta = \zeta_0(\tau, \hat{x}, \hat{y}, \xi) + \varepsilon \zeta_1 + \cdots \tag{18}$$

and equating terms of order ε^0 and ε^1, we may derive the following equations for (F_0, ζ_0) and (F_1, ζ_1):

$$\zeta_0 = \frac{\partial F_0}{\partial \xi}\,; \tag{19a}$$

$$\begin{aligned} -\frac{\partial \zeta_1}{\partial \xi} + \frac{\partial^2 F_1}{\partial \xi^2} = {} & \frac{\partial \zeta_0}{\partial \tau} + 2h^{1/2}\frac{\partial^2 F_0}{\partial \hat{x}\, \partial \xi} + \frac{1}{2}h^{-1/2}\frac{\partial h}{\partial \hat{x}}\frac{\partial F_0}{\partial \xi} + h^{-1}\left(\zeta_0 \frac{\partial^2 F_0}{\partial \xi^2} + \frac{\partial F_0}{\partial \xi}\frac{\partial \zeta_0}{\partial \xi}\right) - \frac{\kappa_0}{6}h\frac{\partial^4 F_0}{\partial \xi^4} \\ & + \frac{\nu_0^2}{\kappa_0}\left[\gamma^2 \frac{\partial}{\partial \hat{y}}\left(h\frac{\partial F_0}{\partial \hat{y}}\right) + \gamma \frac{\partial}{\partial \hat{y}}\left(h\mathscr{G}\frac{\partial F_0}{\partial \xi}\right) + \gamma h \mathscr{G}\frac{\partial^2 F_0}{\partial \hat{y}\, \partial \xi} + h\mathscr{G}^2 \frac{\partial^2 F_0}{\partial \xi^2}\right]; \end{aligned} \tag{19b}$$

$$-\frac{\partial F_1}{\partial \xi} + \zeta_1 = -\frac{1}{2}h^{-1}\left(\frac{\partial F_0}{\partial \xi}\right)^2 - \frac{\kappa_0}{2}h\frac{\partial^3 F_0}{\partial \xi^3}\,. \tag{19c}$$

Again, as expected, the first equation (19a) is insufficient to determine both functions ζ_0 and F_0 and it is necessary to go to second order in ε to obtain a consistency condition to do this.

Differentiating (19c) by $\partial/\partial \xi$ and subtracting from (19b) gives:

$$\begin{aligned} & \frac{\partial \zeta_0}{\partial \tau} + 2h^{1/2}\frac{\partial \zeta_0}{\partial \hat{x}} + \frac{1}{2}h^{-1/2}\frac{\partial h}{\partial \hat{x}}\zeta_0 + 3h^{-1}\zeta_0\frac{\partial \zeta_0}{\partial \xi} + \frac{\kappa_0}{3}h\frac{\partial^3 \zeta_0}{\partial \xi^3} + \frac{\nu_0^2}{\kappa_0}h\mathscr{G}^2\frac{\partial \zeta_0}{\partial \xi} \\ & \quad + \frac{\nu_0^2}{\kappa_0}\gamma h \mathscr{G}\frac{\partial \zeta_0}{\partial \hat{y}} + \frac{\nu_0^2}{\kappa_0}\gamma \frac{\partial}{\partial \hat{y}}\left[\gamma h \frac{\partial F_0}{\partial \hat{y}} + h\mathscr{G}\zeta_0\right] = 0\,, \end{aligned} \tag{20}$$

with

$$F_0 = \int_\infty^\xi \zeta_0\, \mathrm{d}\xi'\,.$$

From the above equation (20), when ζ_0 is *independent* of the slow time τ, we may rederive equation (22) in Xue-Nong Chen's [40] paper.

Naturally, our extended KP equation for an uneven bottom for ζ_0 can be derived directly from the single Boussinesq equation (15).

Furthermore, if topography is even ($h \equiv 1$), equation (20) is reduced to the classical KP equation (4.6) and if $\nu_0 \equiv 0$, equation (20) is reduced to a varying-coefficient KdV equation that is the same (when $\partial\zeta_0/\partial\tau = 0$) as that of Johnson [38].

To get a more concise form of (20), let us take:

$$\zeta_0 = h^{-1/4}\mathcal{H}(\hat{x}, \hat{y}, \xi) , \quad \text{when}\, \frac{\partial\zeta_0}{\partial\tau} = 0 , \tag{21}$$

so that (20) becomes:

$$\frac{\partial\mathcal{H}}{\partial\hat{x}} + \frac{3}{2}h^{-7/4}\mathcal{H}\frac{\partial\mathcal{H}}{\partial\xi} + \frac{\kappa_0}{6}h^{1/2}\frac{\partial^3\mathcal{H}}{\partial\xi^3} + \frac{\nu_0^2}{2\kappa_0}h^{1/2}\mathcal{G}^2\frac{\partial\mathcal{H}}{\partial\xi} + \gamma\frac{\nu_0^2}{2\kappa_0}h^{3/4}\mathcal{G}\frac{\partial}{\partial\hat{y}}(h^{-1/4}\mathcal{H})$$
$$+ \gamma\frac{\nu_0^2}{2\kappa_0}h^{-1/4}\frac{\partial}{\partial\hat{y}}\left[\gamma h\frac{\partial F_0}{\partial\hat{y}} + h^{3/4}\mathcal{G}\mathcal{H}\right] = 0 , \tag{22}$$

with

$$F_0 = h^{-1/4}\int_\infty^\xi \mathcal{H}\, \mathrm{d}\xi' . \tag{23}$$

Again, when $\nu_0 \equiv 0$ it is possible to rederive the classical KdV equation for an uneven bottom in several forms and for this one may consult the book of Mei ([12], pp. 560–561).

7. Conclusions

In this review, we have attempted to highlight some of the major findings regarding the modeling of non-linear water surface waves, with emphasis on the progress made in the past 15 years.

Waves on the free surface of a body of water have always been a fascinating subject, for they represent a familiar yet complex phenomenon, easy to observe but very difficult to describe.

It must be kept in mind that, at the present time, (asymptotic) modeling, i.e., the translation into correctly expressed mathematical terms of a complex physical situation, has become very important in the realm of scientific research in many fields. This is particularly true for fluid mechanics (see, e.g., Zeytounian [54, 55]) and so also for the theory of non-linear water waves, which is considered to be a subdiscipline of fluid mechanics taken as a whole.

But, in this review we have considered only the classical problem for an incompressible, irrotational and inviscid fluid (water). The problem of the account of compressibility and rotational effects from the full Euler equations is another question and for the present time a general modeling of these effects remains unresolved. It is true that from these full Euler equations it is possible to derive a KP equation (with variable coefficients), but paradoxically the derivation of Boussinesq equations is more subtle! Concerning the account of effects of viscosity and thermal conductivity, this problem is more complicated, principally because of the boundary conditions on a free surface. In this case a very interesting problem is the modeling of 'non-linear waves on the free (inclined) surface of a falling liquid film'.

In future work, we shall investigate (revisit!) these important problems on the basis of asymptotic modeling.

Hence, for the moment, it has been shown in this review that the limit procedure

$$\delta^2 = \kappa_0 \varepsilon\,, \qquad \Delta = \nu_0 \varepsilon\,, \quad \varepsilon \to 0\,,$$

applied to a full water-wave dimensionless potential problem (for a mathematical formulation of the physical water waves problem, see Chapter 13 in the book by Whitham [8]) yields, first, the quasi-one-dimensional generalized Boussinesq equations for the free-surface position and the value of the velocity-potential on the flat bottom (see equation (3.8)). These last equations are *correct up* to order $O(\varepsilon^3)$ inclusive.

From these Q1DGB equations we derived, also in Section 3, the properly so-called quasi-one-dimensional Boussinesq single equation (3.15), with an error of $O(\varepsilon^2)$. Again, with an error of $O(\varepsilon^2)$, the system of Q1DB equations for the free surface position and the horizontal velocities components can be written as in (3.20) or (3.21).

Next, in Section 4, the KP equation (4.6) and the associated linear inhomogeneous equation (4.13) are derived, describing the second-order 'KP approximation'. As an interesting consequence of (4.13), it is possible to introduce the notion of a 'dressed KP soliton', that is the KP soliton involving second-order corrections.

In most cases of interest in water waves We < 1/3 and consequently, the KP equation (4.6) may be integrated to

$$2\frac{\partial F_0}{\partial \tau} + \frac{3}{2}\left(\frac{\partial F_0}{\partial \xi}\right)^2 + \kappa_0\left(\frac{1}{3} - \mathrm{We}\right)\frac{\partial^3 F_0}{\partial \xi^3} = \frac{\nu_0^2}{\kappa_0}\int_{\xi}^{\infty} \frac{\partial^2 F_0}{\partial y^2}\,\mathrm{d}\xi'\,,$$

and this equation is now of the form of an evolution equation for F_0.

But, for very thin sheets of water (i.e. We large enough), this last KP equation is false, the long water waves travel slower than their neighbors, and the integral should be from $-\infty$ to ξ.

In Section 5, the evolution of slowly modulated weakly non-linear water waves is analyzed. First, we derive a 'generalized' KP equation (5.13a), with (5.13b), and obtain for the long-wave, not only the classical coupled Schrödinger–Poisson system of two equations (5.37) and (5.38), for the amplitude of a progressive wave-packet of small amplitude and the mean part of the velocity-potential of the flow, but also a third complementary equation (5.48) for the term proportional to α in the expansion of velocity-potential (see (5.50)), when $\alpha \to 0$. As a result we can calculate the expansion of the free-surface position up to order $O(\alpha^2)$ inclusive (see (5.49)) and also the expansion for the velocity-potential up to order $O(\alpha)$ inclusive, when $\alpha \to 0$.

Next, we confirm that the long-wave limit ($\delta \to 0$) of the full non-linear Schrödinger–Poisson coupled system of two equations: (5.61), (5.62), matches precisely the short-wave limit of the KP equation (4.6) (when $\kappa_0 \to \infty$).

The last section, Section 6, is devoted to the influence of an uneven bottom. For the derivation of integrable non-linear equations for water-waves in straits of varying depth and width and also for the solitons in shallow seas of variable depth and in marine straits, see the papers by David et al. ([56, 57]).

In the present paper the various forms of Q1DB-equations for a variable depth (see for example, equations (6.6), (6.7) and (6.14) or (6.15)) are derived. From the equations (6.6) and (6.7) we derive also an extended KP equation (6.20), for an uneven bottom. Naturally, in principle, from this last KP equation (6.20) it is also possible to derive a non-linear Schrödinger–Poisson system of two equations for variable depth, when $\kappa_0 \to \infty$, but this derivation would be a very tedious task!

In the *deep-water limit*:

$$\delta \to \infty \quad \text{but} \quad \varepsilon\delta \ll 1 ,$$

and We fixed, we find that the equation for ϕ_{00}, see (5.61), is homogeneous and always of elliptic type and the solution then reduces simply to

$$\frac{\partial \phi_{00}}{\partial \hat{q}} = 0 , \qquad \frac{\partial \phi_{00}}{\partial \hat{\eta}} = 0 .$$

In this case, equation (5.62), for A, becomes

$$i \frac{\partial A}{\partial \hat{\tau}} + \lambda_\infty \frac{\partial^2 A}{\partial \hat{q}^2} + \mu_\infty \frac{\partial^2 A}{\partial \hat{\eta}^2} = \chi_\infty A|A|^2 ,$$

and the long-wave/short-wave resonance has disappeared. This last Schrödinger equation was first derived in the context of deep-water by Zakharov [29]. Since the relative signs of the two-dispersive terms are opposite ($\lambda_\infty < 0$, if $1 - 6\,\mathrm{We} - 3\,\mathrm{We} > 0$, when $\mathrm{We} \neq 0$, but $\mu_\infty > 0$), then this equation is hyperbolic in the spatial variables. We note also that the second-harmonic resonance is still present as is manifested by the factor $1/(1-2\,\mathrm{We})$ in the coefficient χ_∞, in the non-linear term.

For a modification of this Zakharov–Schrödinger equation (new effect introduced to order ε^4), see Dysthe [58]. Concerning the non-linear dynamics of deep-water gravity waves, see the review article by Yuen and Lake [30].

It is interesting also to examine the appropriate *near-field* of the KP equation and thereby derive the general form of the initial value problem for this KP equation. It is possible to find the precise nature of a near-field which will match the far-field (when $\tau = \varepsilon t$), although a neighbourhood of the origin must be excluded!

The KP equation (4.6) can be examined as $\tau \to 0$, by considering equation (3.15) for $F^*(x, y, t)$, when $\varepsilon \to 0$ with x, y and t fixed, whence:

$$\frac{\partial^2 F^*}{\partial t^2} - \frac{\partial^2 F^*}{\partial x^2} = 0 ,$$

to leading order.

Thus for right-travelling waves, we have: $F^* \sim \mathscr{F}(x - t, y)$ where $\mathscr{F}$ is an arbitrary function [related to ζ^0; see (2.5)], and matching from the far-field ($t \to +\infty \sim \tau \to 0$) is therefore possible if (see the equation (4.6)):

$$F_0 \to \mathscr{F}(\xi, y) , \quad \text{as } \tau \to 0 .$$

Concerning the *rigorous* results (well-posed nature of the initial value problem, existence and uniqueness) for the Boussinesq, KdV and NLS equations, the reader can consult the paper by Craig and Sulem and Sulem [59]. In the book by Shinbrot ([60], p. 87) the reader can find also information concerning the first results of the proof of existence of a solution for the water-wave problem.

The paper by Debnath [61] is devoted to theoretical developments in *bifurcation* and *non-linear instability problems* in applied mathematics and concerning the water-waves problem (non-linear Rayleigh–Taylor and Kelvin–Helmoltz instability, instability and bifurcation of non-linear wavetrain) see in this paper, Sections 6.1, 7.1 and 13.1–13.4. The book by Craik [35] is a comprehensive account of theory and experiment on *water-wave interaction* phenomena and phenomena of non-linear hydrodynamic stability, especially those leading to the *onset of turbulence*. Finally, in the book by Infeld and Rowlands ([26], Chap. 10) the reader can find some information concerning '*non-coherent phenomena*'.

It should also be noted that most of the perturbative work presented here is closely connected to the general presentation by Zakharov, Calogero and Eckhaus (see the corresponding references below).

Finally, we find that for the initial-value water-waves problem, for the velocity potential ϕ, some degree of mathematical *intractability* seems inevitable here (see Benjamin [62]). We recognize the probability that the *general initial-value problem cannot be correctly posed* (well set), because it is known that in practice *water waves may break*! – that is, the (*rotational*) motion may become *turbulent* and so *lose continuous dependence on initial data* (in this case the *emergence of chaos via a strange attractor* is possible). This aspect of the subject still remains largely *mysterious*, and reservations regarding it are needed to put any theoretical work on water waves into a proper scientific perspective.

The fact that most existing theory – linearized, long waves and weakly non-linear approximations – is essentially tentative does not, of course, impair its practical value.

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I ask for the indulgence of English-speaking readers, thinking that they might prefer a text in not 'quite perfect' English rather than in 'perfect' French! Indeed, the most important question, from my point of view, concerns mainly the scientific content of the present paper and I hope that in this way the readers will be satisfied.

Notes

* Equation (2.11) is equation (11) of Section 2.

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